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THE EFFECT OF PRICE CHANGES ON THE DEMAND AND SUPPLY OF SEPARABLE GOODS*

BY MASAZO SONO

1. STATEMENT OF THE PROBLEM

IN THIS PAPER I shall examine the effect of a change in the price of one commodity on the quantities of "separable goods" exchanged in equilibrium. I was led to this inquiry by doubts about Hicks' definitions of substitutability and complementarity among commodities.[1] He divided the change in the demand and supply of one commodity, which results from a change in price of another commodity, into an income and a substitution effect. Hicks called the two terms in his equation which measure these effects the "income term" and the "substitution term," respectively, and distinguished substitutability and complementarity between the two commodities by the positive or negative sign of the latter term.[2]

The following questions then arose: (i) Is it not possible that two commodities, originally substitutes for each other, may become complementary when the number of commodities in the preference field changes? If so, what causes this phenomenon? (ii) Hicks did not specify the good undergoing a price change. The same definition, therefore, might well be applied to an independent good; is this ad-

* Professor Masazo Sono's paper on the separability of goods in the theory of consumer behavior was first published in Japanese in *Kokumin Keizai Zasshi*, LXXIV (1945), 1-51. This work anticipates similar studies published later in the English language. The editors consider Sono's paper to be a work of great merit in economic theory, unfortunately not available to English language readers. We are, therefore, publishing it at this time in translation, together with a historical note on the subject matter by Professor Morishima. The translating of this paper was supported by the Ford Foundation as part of a project for promoting the translation of Japanese economic studies. The translator is K. Sato of Osaka University.

[1] J. R. Hicks, *Value and Capital*, (Oxford: Oxford Univ. Press, 1939); *Theorie mathématique de la valeur*, (Paris: Hermann, 1937).

[2] In the 1937 book Hicks stated his definitions in terms of elasticities, with analogous criteria for substitutability and complementarity.

missible? (iii) According to Hicks, the sign of the substitution term largely determines the sign of the total effect of a price change on demand. How strong can this influence be?

These were my original questions. I have somewhat expanded the problem in order to examine the effect of price changes on the demand and supply of separable goods. The present inquiry will try to ascertain the suitability of Hicks' definitions of substitutability and complementarity.

Let us now give a brief, if incomplete, definition of "separable goods." Commodities A, B, C, etc. are called "separable" from commodity X if the order of the values of preference (i.e., the Pareto indices) corresponding to various amounts of commodities A, B, C, etc. is independent of the quantity of commodity X.

Assume that a number of commodities which are at exchange equilibrium for an individual can be divided into two groups, and that the commodities in the second group are separable from those in the first. If all or some of the commodities in the first group are excluded from the preference field, there will be changes in the equilibrium quantities of the commodities in the second group. Money income can, however, be so adjusted as to leave the equilibrium quantities unchanged both before and after the exclusion of the first-group commodities. (See Section 2.4.) This I call the "point of common equilibrium." On the basis of the existence of this point of common equilibrium, I have examined the effect of a change in the price of one commodity on the equilibrium quantities of separable goods and have derived certain "laws of proportionality," which are given below.

Let the adjective "general" denote the case where commodities in the first group are included in the preference field, and the adjective "proper" denote the case where they are excluded. Then we have the following propositions:

(1) The "general" propensities to consume separable goods are proportional to the respective "proper" propensities to consume them (at the point of common equilibrium), (see Section 2.5 below), where the propensity to consume X is the rate of change in the equilibrium quantity of X with respect to changes in money income, i.e., the differential coefficient of the equilibrium quantity of X with respect to money income.

(2) When there is a change in the price of one of the (separable) commodities of the second group, the difference between the "general" and "proper" effects on the equilibrium quantity of each separable good is proportional to the "proper" propensity to consume it (at the

point of common equilibrium). This ratio is equal to the rate of change in the total expenditure on commodities of the second group as a whole with respect to the change in price of a separable commodity. (See Section 3.8.)

(3) Similarly, the difference between the "general" and "proper" substitution terms is proportional to the "proper" propensity to consume (at the point of common equilibrium). (See Section 3.8.)

The substitution term will in general differ, possibly even in sign as the example in Section 3.9 illustrates, according as the commodities of the first group are present or absent from the preference field. We must thus reconsider the suitability of Hicks' defining substitutability and complementarity according to the sign of the substitution term.

In examining the substitution effect, Hicks considered the substitution of goods which leaves the value of the "general" preference function unchanged. But if we make a slight change in our definition of "substitution" and consider a substitution of goods such that the value of the "proper" preference function remains unchanged, then the substitution effect in this case always coincides with the "proper" substitution effect (commodities of the first group being excluded), whether or not the commodities of the first group exist. (See Section 3.6.) The sum of the "proper" income effect and the "proper" substitution effect is equal to the change in the equilibrium quantity. In other words, the "general" substitution effect can be divided into two parts, one of which is the "proper" substitution effect; the other, together with the "general" income effect, constituting the "proper" income effect. (See Section 3.7.)

(4) When the price of a commodity belonging to the first group changes, the rate of change in the equilibrium quantity of a separable good belonging to the second group and its substitution term are proportional to the "proper" propensity to consume this separable good (at the point of common equilibrium). The ratio of the rate of change in equilibrium quantity to the "proper" propensity to consume equals the rate of change in expenditure on commodities of the second group as a whole with respect to the change in price of a commodity in the first group. (See Section 3.4.)

The proportions listed above are not necessarily positive. In (4), for example, the effect of a price change on the equilibrium quantity of a separable good and its substitution term are proportional to each other, but the proportion need not be positive. Hence, the two may take the same or opposite signs for all the commodities in the second

group.

When the price of a commodity independent of the other commodities changes, it is very simple to ascertain the signs of the proportion.[3] We have then the following results:

(5) The substitution term of each commodity (except the commodity undergoing a price change) has the same sign as the "proper" propensity to consume it (at the point of common equilibrium). (See equation (54) below.) The substitution term of a superior good is always positive, while that of an inferior good is always negative, as long as the commodity whose price changes is independent of other goods. (The definition of a "superior" or "inferior" good presumes the exclusion of the independent good.) This conclusion demonstrates that Hicks' definitions of substitutability and complementarity according to the sign of the substitution term cannot be applied to independent goods.

(6) The effect of a change in the price of one good on the equilibrium quantity of every other good has a sign opposite to the "proper" propensity to consume it (at the point of common equilibrium) if the elasticity of the rate of substitution of the aforementioned good for each other good is greater than unity; it has the same sign if the elasticity of rate of substitution is less than unity. (See equation (53).)

(7) The rate of change in expenditure on an independent good in response to a change in its own price is positive or negative according as the above mentioned elasticity is greater or less than unity. (See equation (51).) Here the definition of the rate of substitution is Hicks', and elasticity of substitution is defined in the usual way. In the case of an independent good, this elasticity is determined solely by the equilibrium quantity of the independent good, and the equilibrium quantities of other goods are irrelevant. Therefore, the elasticity of the rate of substitution for any other good takes the same value. (See Section 5.1.)

2. SEPARABILITY OF GOODS

2.1. *The Definition of Separable Goods.* Let $x_1, x_2, \cdots, x_n$ denote quantities of goods $X_1, X_2, \cdots, X_n$, and $\varphi(x_1, x_2, \cdots, x_n)$ an individual's preference function. We shall call the partial derivative $\partial\varphi/\partial x_i[= \varphi_i]$

[3] By an independent good we mean a good such that its marginal degree of preference (the partial derivative of the preference function with respect to this good) is independent of other goods. The definition in no way implies that the *use* of an independent good is completely unrelated to the *use* of other goods. The definition is derived only from certain properties of the preference function. (See Section 4.1.)

the *marginal degree of preference* of good X_i and $\varphi_i dx_i$ the *marginal value of preference*. The ratios of the marginal degree of preference

$$\varphi_1 : \varphi_2 : \cdots : \varphi_n$$

are invariant with respect to the choice of function φ, given the quantities of goods, but are not independent of the unit of measurement of the xs. The ratios of the marginal values of preference, however,

$$\varphi_1 dx_1 : \varphi_2 dx_2 : \cdots : \varphi_n dx_n$$

are invariant with respect to changes in the unit of measurement as well as with respect to changes in the function φ chosen.

When the ratios of the marginal values of preference for any given quantities of commodities, $X_{l+1}, X_{l+2}, \cdots, X_n$ are independent of the quantity of another commodity $X_i (1 \leqq i \leqq l)$, i.e. when

$$\frac{\partial}{\partial x_i}\left(\frac{\varphi_j dx_j}{\varphi_k dx_k}\right) = 0 , \qquad \left.\begin{matrix} j \\ k \end{matrix}\right\} = l+1,\ l+2, \cdots, n ,$$

commodities $X_{l+1}, X_{l+2}, \cdots, X_n$ are called *separable* from commodity X_i. The necessary and sufficient conditions for separability are

$$(1) \qquad \frac{\varphi_{l+1,i}}{\varphi_{l+1}} = \frac{\varphi_{l+2,i}}{\varphi_{l+2}} = \cdots = \frac{\varphi_{ni}}{\varphi_n} , \qquad \left(\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right),$$

since

$$\frac{\partial}{\partial x_i}\left(\frac{\varphi_j dx_j}{\varphi_k dx_k}\right) = \frac{(\varphi_{ji}\varphi_k - \varphi_{ki}\varphi_j) dx_j}{\varphi_k^2 dx_k} .$$

Denoting the value of the ratios (1) by ν_i, we obtain

$$(2) \qquad \varphi_{l+r,i} = \nu_i \varphi_{l+r} , \qquad r = 1, 2, \cdots, n-l ;$$

ν_i is, in general, a function of $x_1, x_2, \cdots, x_n$.

2.2. *Properties of Separable Goods.* PROPOSITION 1. *Let there be two sets of quantities of the commodities*

$$(3) \qquad \begin{aligned} &(x_1', x_2', \cdots, x_l', x_{l+1}', \cdots, x_n') , \\ &(x_1', x_2', \cdots, x_l', x_{l+1}'', \cdots, x_n'') , \end{aligned}$$

such that

$$(4) \quad \varphi(x_1', x_2', \cdots, x_l', x_{l+1}', \cdots, x_n') = \varphi(x_1', x_2', \cdots, x_l', x_{l+1}'', \cdots, x_n'') .$$

If commodities $X_{l+1}, X_{l+2}, \cdots, X_n$ are separable from X_i $(i = 1, 2, \cdots, l)$, then this equality always holds whatever values X_i may take.

PROOF: By (2),

$$(5)\qquad \frac{\partial}{\partial x_i}\left\{\sum_{r=1}^{n-l}\varphi_{i+r}dx_{i+r}\right\}=\sum_{r=1}^{n-l}\varphi_{i+r,i}dx_{i+r}=\nu_i\left\{\sum_{r=1}^{n-l}\varphi_{i+r}dx_{i+r}\right\},$$

hence

$$\frac{\partial}{\partial x_i}\left\{\sum_{r=1}^{n-l}\varphi_{i+r}dx_{i+r}\right\}=0 \qquad \text{if} \qquad \sum_{r=1}^{n-l}\varphi_{i+r}dx_{i+r}=0.^{4}$$

PROPOSITION 2. *For a given set of quantities of commodities* $X_1, X_2, \cdots, X_i$, $(x_1', x_2', \cdots, x_i')$, *choose any two sets of quantities of commodities* $X_{i+1}, X_{i+2}, \cdots, X_n$,

$$(x_{i+1}', x_{i+2}', \cdots, x_n') \quad \text{and} \quad (x_{i+1}'', x_{i+2}'', \cdots, x_n''),$$

which satisfy equality (4). *If this equality holds for any amount of commodity* X_i, *then commodities* $X_{i+1}, X_{i+2}, \cdots, X_n$ *are separable from* X_i.

The proof is omitted. The converse of Proposition 1 also holds. This allows us to adopt Proposition 1 as the definition of separable goods.

PROPOSITION 3. *Let there be two sets of quantities* (3), *such that*

$$(6)\qquad \varphi(x_1', x_2', \cdots, x_i', x_{i+1}', \cdots, x_n') > \varphi(x_1', x_2', \cdots, x_i', x_{i+1}'', \cdots, x_n'').$$

If commodities $X_{i+1}, X_{i+2}, \cdots, X_n$ *are separable from* X_i $(i = 1, 2, \cdots, l)$, *then this inequality always holds whatever values commodity* X_i *may take.*

PROOF: To simplify notation, let $i = 1$. Assume, for a certain value x_1'' of x_1, that

$$\varphi(x_1'', x_2', \cdots, x_i', x_{i+1}', \cdots, x_n') \leqq \varphi(x_1'', x_2', \cdots, x_i', x_{i+1}'', \cdots, x_n'').$$

Then, when the value of x_1 moves from x_1' to x_1'', it must pass through an intermediate value $\bar{x}_1$ such that

$$\varphi(\bar{x}_1, x_2', \cdots, x_i', x_{i+1}', \cdots, x_n') = \varphi(\bar{x}_1, x_2', \cdots, x_i', x_{i+1}'', \cdots, x_n'').$$

It follows then from Proposition 1 that inequality (6) should be an equality, contrary to our assumption.

2.3. *The "Proper" Preference Function.* Let n commodities be divided into two groups, and assume that commodities $X_{i+1}, X_{i+2}, \cdots, X_n$ of the second group are separable from each of the commodities $X_1, X_2, \cdots, X_i$ of the first group. Then, according as

[4] Throughout this paper it is assumed that the preference function is continuous everywhere and that its first and second partial derivatives are also continuous.

$$\varphi(x_1', x_2', \cdots, x_i', x_{i+1}', \cdots, x_n') \gtreqless \varphi(x_1', x_2', \cdots, x_i', x_{i+1}'', \cdots, x_n'')$$

for the two sets of quantities in (3),

$$(7) \quad \varphi(x_1, x_2, \cdots, x_i, x_{i+1}', \cdots, x_n') \gtreqless \varphi(x_1, x_2, \cdots, x_i, x_{i+1}'', \cdots, x_n'')$$

for any values of $x_1, x_2, \cdots, x_i$, as a consequence of Propositions 1 and 3. Thus, if fixed values $a_1, a_2, \cdots, a_i$ are assigned to $x_1, x_2, \cdots, x_i$ of the first group, and if we put

$$(8) \qquad \eta(x_{i+1}, \cdots, x_n) = \varphi(a_1, a_2, \cdots, a_i, x_{i+1}, \cdots, x_n) ,$$

we obtain

$$\eta(x_{i+1}', \cdots, x_n') \gtreqless \eta(x_{i+1}'', \cdots, x_n'')$$

from the inequality (7) and *vice versa*.

We can let η represent the preference function of the commodities $X_{i+1}, X_{i+2}, \cdots, X_n$ of the second group when the commodities of the first group are excluded from the preference field. We call this the *"proper" preference function* (of $X_{i+1}, X_{i+2}, \cdots, X_n$).

We can now express φ as a function of η and $x_1, x_2, \cdots, x_i$. Let $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_i, \bar{\eta}$ be an arbitrary set of values of $x_1, x_2, \cdots, x_i, \eta$, and let

$$\eta(x_{i+1}', \cdots, x_n') = \bar{\eta} , \quad \text{and} \quad \eta(x_{i+1}'', \cdots, x_n'') = \bar{\eta} .$$

Then, from (8),

$$\varphi(a_1, \cdots, a_i, x_{i+1}', \cdots, x_n') = \varphi(a_1, \cdots, a_i, x_{i+1}'', \cdots, x_n'') ,$$

and, therefore,

$$\varphi(\bar{x}_1, \cdots, \bar{x}_i, x_{i+1}', \cdots, x_n') = \varphi(\bar{x}_1, \cdots, \bar{x}_i, x_{i+1}'', \cdots, x_n'') .$$

Only one value of φ is determined corresponding to the arbitrary set of values $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_i, \bar{\eta}$; therefore, φ is a function of these variables.

We can thus conceive of a "proper" preference function for commodities $X_{i+1}, X_{i+2}, \cdots, X_n$, which are separable from each of the commodities $X_1, X_2, \cdots, X_i$; we can then construct a "general" preference function from the "proper" one and the quantities of the other commodities $X_1, X_2, \cdots, X_i$. This, however, is possible only when commodities are separable.

2.4. *Point of Common Equilibrium.* Let $p_1, p_2, \cdots, p_n$ denote the prices of commodities $X_1, X_2, \cdots, X_n$; $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote the individual's initial holdings of these commodities; and α denote his initial stock of money. We take for the budget equation,

$$(9) \qquad p_1(x_1 - \alpha_1) + p_2(x_2 - \alpha_2) + \cdots + p_n(x_n - \alpha_n) = \alpha .$$

The conditions for a maximum of the preference function subject to the condition (9) are

$$\frac{\varphi_1}{p_1} = \frac{\varphi_2}{p_2} = \cdots = \frac{\varphi_n}{p_n} . \tag{10}$$

These are the equations of individual exchange equilibrium and, together with (9), determine the equilibrium quantities of commodities exchanged by the individual[5]. Denoting the value of the ratios (10) by λ, we obtain

$$\varphi_i = \lambda p_i , \qquad i = 1, 2, \cdots, n . \tag{11}$$

We assume, as before, that commodities $X_{l+1}, X_{l+2}, \cdots, X_n$ of the second group are separable from commodities $X_1, X_2, \cdots, X_l$ of the first group. The preference function φ can then be expressed as a function of the "proper" preference function η and of the quantities of the commodities of the first group, i.e.,

$$\varphi = f(x_1, x_2, \cdots, x_l, \eta) .$$

For $j > l$,

$$\varphi_j = \eta_j \frac{\partial f}{\partial \eta} , \qquad \text{where} \quad \eta_j = \frac{\partial \eta}{\partial x_j} .$$

Hence, the latter part of equations (10),

$$\frac{\varphi_{l+1}}{p_{l+1}} = \frac{\varphi_{l+2}}{p_{l+2}} = \cdots = \frac{\varphi_n}{p_n} ,$$

can be rewritten as

$$\frac{\eta_{l+1}}{p_{l+1}} = \frac{\eta_{l+2}}{p_{l+2}} = \cdots = \frac{\eta_n}{p_n} . \tag{12}$$

In other words, the marginal degree of preference of each commodity in the proper preference function is proportional to its price at the point of equilibrium.

Let the quantities of commodities $X_1, X_2, \cdots, X_n$ exchanged in equilibrium [i.e., the solutions of simultaneous equations (9) and (10)] be $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$, and put

$$\alpha - p_1(\bar{x}_1 - \alpha_1) - p_2(\bar{x}_2 - \alpha_2) - \cdots - p_l(\bar{x}_l - \alpha_l) = m , \tag{13}$$

[5] Throughout the present paper we asseme that the preference function has indifference surfaces convex to the origin, i.e., that the stability conditions are always satisfied in Hicks' sense for the equilibrium of individual exchange. Also, unless otherwise stated, the marginal value of preference is not assumed to be constant.

where m is the total expenditure on the commodities of the second group. Obviously, the values

$$\bar{x}_{l+1}, \bar{x}_{l+2}, \cdots, \bar{x}_n \tag{14}$$

satisfy the equation

$$p_{l+1}(x_{l+1} - \alpha_{l+1}) + p_{l+2}(x_{l+2} - \alpha_{l+2}) + \cdots + p_n(x_n - \alpha_n) = m \tag{15}$$

and (12) simultaneously. In other words, the values (14) of $x_{l+1}, x_{l+2}, \cdots, x_n$ maximize the proper preference function η subject to the condition (15). Spelled out, this means that the values (14) show the equilibrium quantities of commodities of the second group, when all the commodities of the first group are excluded from the preference field, and only commodities of the second group are left to choice, m being the initial holding of money. Thus we find that even if some—not all—of the commodities of the first group, from which the second group are separable, were excluded from a consumer's preference field, he would buy the same quantities of separable commodities as before if there is a proper adjustment of his initial holding of money.[6] This property plays a basic role in the examination of the effect of price changes on the equilibrium quantities of separable goods.

2.5. *The Propensity to Consume Separable Goods: The Law of Proportionality.* Let us call $\partial x_j/\partial \alpha$ the propensity to consume X_j with respect to money income. If the initial holding of commodity X_k alone changes, the marginal propensity will be denoted as $\partial x_j/\partial \alpha_k$. It is obvious that the relationship between these two marginal propensities is

$$\frac{1}{p_k}\frac{\partial x_j}{\partial \alpha_k} = \frac{\partial x_j}{\partial \alpha}, \qquad j = 1, 2, \cdots, n. \tag{16}$$

We assume, as before, that the commodities of the second group are separable from those of the first group. We then compare the propensities to consume commodities of the second group at the point of common equilibrium both with and without the commodities of the

[6] Take

$$\eta = \varphi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_l, x_{l+1}, \cdots, x_n)$$

for the "proper" preference function of the commodities of the second group. Then, at the point of common equilibrium,

$$\eta_i(\bar{x}_{l+1}, \bar{x}_{l+2}, \cdots, \bar{x}_n) = \varphi_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n),$$
$$\eta_{ij}(\bar{x}_{l+1}, \bar{x}_{l+2}, \cdots, \bar{x}_n) = \varphi_{ij}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n), \qquad i, j = l+1, l+2, \cdots, n.$$

It follows directly that the stability conditions are satisfied by the "proper" preference function at the point of common equilibrium if they are satisfied by the "general" preference function.

first group in the preference field. First, we rewrite equation (9) as

$$p_1(x_1-\alpha_1)+p_2(x_2-\alpha_2)+\cdots+p_l(x_l-\alpha_l)=\beta\,, \tag{17}$$

$$p_{l+1}(x_{l+1}-\alpha_{l+1})+p_{l+2}(x_{l+2}-\alpha_{l+2})+\cdots+p_n(x_n-\alpha_n)=\gamma\,, \tag{18}$$

where $\beta+\gamma=\alpha$. Differentiating equation (18) and the general equilibrium equations

$$\varphi_{l+r}=\lambda p_{l+r}\,, \qquad r=1,2,\cdots,n-l\,, \tag{19}$$

with respect to α, we obtain

$$p_{l+1}\frac{\partial x_{l+1}}{\partial\alpha}+p_{l+2}\frac{\partial x_{l+2}}{\partial\alpha}+\cdots+p_n\frac{\partial x_n}{\partial\alpha}=\frac{\partial\gamma}{\partial\alpha}\,, \tag{20}$$

$$\begin{aligned}-p_{l+r}\frac{\partial\lambda}{\partial\alpha}+\varphi_{l+r,1}\frac{\partial x_1}{\partial\alpha}+\varphi_{l+r,2}\frac{\partial x_2}{\partial\alpha}+\cdots+\varphi_{l+r,l}\frac{\partial x_l}{\partial\alpha}\\ +\varphi_{l+r,l+1}\frac{\partial x_{l+1}}{\partial\alpha}+\cdots+\varphi_{l+r,n}\frac{\partial x_n}{\partial\alpha}=0\,,\\ r=1,2,\cdots,n-l\,.\end{aligned} \tag{21}$$

Since the commodities of the second group are separable from those of the first group, we also obtain from (2)

$$\varphi_{l+r,i}=\nu_i\varphi_{l+r}\,, \qquad r=1,2,\cdots,n-l, \text{ and } i=1,2,\cdots,l\,. \tag{22}$$

Transforming the first line of equation (21) by (19) and (22), we obtain

$$\begin{aligned}-p_{l+r}\Big(\frac{\partial\lambda}{\partial\alpha}-\lambda\sum_{j=1}^{l}\nu_j\frac{\partial x_j}{\partial\alpha}\Big)+\sum_{s=1}^{n-l}\varphi_{l+r,l+s}\frac{\partial x_{l+s}}{\partial\alpha}=0\,,\\ r=1,2,\cdots,n-l\,.\end{aligned} \tag{23}$$

To examine the *proper* propensities to consume, i.e., the propensities to consume when the commodities of first group are excluded from the preference field, we take

$$\eta=\varphi(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_l,x_{l+1},\cdots,x_n)$$

(as given in footnote 6) for the "proper" preference function of the commodities of the second group, m in (13) for money income, and μ for the value of the ratios (12), i.e.

$$\eta_{l+r}=\mu p_{l+r}\,, \qquad r=1,2,\cdots,n-l\,. \tag{24}$$

Differentiating (15) and (24) with respect to m, we obtain

$$p_{l+1}\left[\frac{\partial x_{l+1}}{\partial m}\right] + p_{l+2}\left[\frac{\partial x_{l+2}}{\partial m}\right] + \cdots + p_n\left[\frac{\partial x_n}{\partial m}\right] = 1\,,$$

$$-p_{l+r}\left[\frac{\partial \mu}{\partial m}\right] + \sum_{s=1}^{n-l} \eta_{l+r,l+s}\left[\frac{\partial x_{l+s}}{\partial m}\right] = 0\,, \qquad r = 1, 2, \cdots, n-l\,.$$

where the "proper" propensities to consume are shown in brackets to distinguish them from the "general" propensities to consume. Since

$$\eta_{l+r,l+s} = \varphi_{l+r,l+s}$$

at the point of common equilibrium; we can write the preceding equations as

$$p_{l+1}\left[\frac{\partial x_{l+1}}{\partial m}\right] + p_{l+2}\left[\frac{\partial x_{l+2}}{\partial m}\right] + \cdots + p_n\left[\frac{\partial x_n}{\partial m}\right] = 1\,,$$

(25)

$$-p_{l+r}\left[\frac{\partial \mu}{\partial m}\right] + \sum_{s=1}^{n-l} \varphi_{l+r,l+s}\left[\frac{\partial x_{l+s}}{\partial m}\right] = 0\,, \qquad r = 1, \cdots, n-l\,.$$

From the stability conditions we know that the determinant composed of the coefficients of these simultaneous equations is nonzero. Therefore, a comparison of (25) with the simultaneous equation systems (20) and (23) readily produces the relations

$$\frac{\dfrac{\partial x_{l+r}}{\partial \alpha}}{\dfrac{\partial \gamma}{\partial \alpha}} = \left[\frac{\partial x_{l+r}}{\partial m}\right]\,, \qquad \frac{\dfrac{\partial \lambda}{\partial \alpha} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial \alpha}}{\dfrac{\partial \gamma}{\partial \alpha}} = \left[\frac{\partial \mu}{\partial m}\right]\,,$$

which in turn lead to

$$(26)\quad \frac{\dfrac{\partial x_{l+1}}{\partial \alpha}}{\left[\dfrac{\partial x_{l+1}}{\partial m}\right]} = \frac{\dfrac{\partial x_{l+2}}{\partial \alpha}}{\left[\dfrac{\partial x_{l+2}}{\partial m}\right]} = \cdots = \frac{\dfrac{\partial x_n}{\partial \alpha}}{\left[\dfrac{\partial x_n}{\partial m}\right]} = \frac{\dfrac{\partial \lambda}{\partial \alpha} - \lambda \sum_{j=1}^{l} \nu_j \dfrac{\partial x_j}{\partial \alpha}}{\left[\dfrac{\partial \mu}{\partial m}\right]} = \frac{\partial \gamma}{\partial \alpha}\,.$$

We can thus state a law of proportionaliy of propensities to consume separable goods: *The "general" propensities to consume separable goods $X_{l+1}, X_{l+2}, \cdots, X_n$ are proportional to the "proper" propensities to consume them (at the point of common equilibrium).*[7]

It must be noted that the "general" and "proper" propensities to consume a commodity have the same sign if the ratio $\partial\gamma/\partial\alpha$ is positive in the equations (26), while they differ in sign if $\partial\gamma/\partial\alpha$ is negative. γ is the total expenditure (or the total revenue, if negative) on the

[7] The law of proportionality also holds when only some of the commodities of the first group are excluded from the preference field.

commodities of the second group as a whole. $\partial\gamma/\partial\alpha$ represents, therefore, the maginal rate of expenditure on separable goods, i.e., the rate of change in expenditure with respect to a change in money income. Thus, if the marginal rate of expenditure is positive, a good which is superior or inferior in one case will remain so in the other. But if it is negative, a superior good in one case will become inferior in the other and *vice versa*.

3. THE EFFECT OF PRICE CHANGES AND THE LAWS OF PROPORTIONALITY

3.1. *The Slutsky Equation.* The effect of changes in the price of one commodity on demand and supply of other commodities can be studied by examining the sign of each term in the Slutsky equations

$$\frac{\partial x_j}{\partial p_k} = -(x_k - \alpha_k)\frac{\partial x_j}{\partial \alpha} + \sigma_{jk}\,, \tag{27}$$

$$\frac{\partial \lambda}{\partial p_k} = -(x_k - \alpha_k)\frac{\partial \lambda}{\partial \alpha} + \sigma_{0k}\,. \tag{28}$$

The first term on the right of (27) shows the effect on equilibrium quantities of X_j of a change in money income, α, with prices held constant. Hicks called this term the *income* term. He called the second term on the right the *substitution* term, and defined goods X_j and X_k to be substitutes if $\sigma_{jk} > 0$ and complements if $\sigma_{jk} < 0$. We obtain

$$\sigma_{jk} = \frac{\lambda D_{k+1,j+1}}{D}\,, \qquad \text{and} \qquad \sigma_{0k} = \frac{\lambda D_{k+1,1}}{D}\,,$$

where

$$D = \begin{vmatrix} 0 & p_1 & \cdots & p_n \\ p_1 & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ p_n & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix},$$

and D_{ij} denotes the cofactor of the i-th column, j-th row element.

3.2. *The Income Term and the Income Effect.* In order to distinguish various equilibrium positions, we shall call the equilibrium determined by equations (9) and (11) the *initial* equilirium. The equilibrium reached when the price of a commodity X_k has been subjected to an infinitesimal change dp_k, all other prices and all initial holdings of goods and money remaining unchanged, is called the *second*

equilibrium. The change in the equilibrium quantity of X_j, when the equilibrium shifts from the *initial* to the *second* position, is given by

$$[\text{i}] \qquad dx_j = -(x_k - \alpha_k)\frac{\partial x_j}{\partial \alpha}\, dp_k + \sigma_{jk} dp_k\,, \qquad i = 1, 2, \cdots, n\,.$$

For the *third* equilibrium we consider the equilibrium reached when money income and initial holdings of goods make infinitesimal changes to

$$\alpha + \delta\alpha,\ \alpha_1 + \delta\alpha_1, \cdots, \alpha_n + \delta\alpha_n\,,$$

while all prices remain unchanged. The equilibrium quantity of X_j changes to $x_j + \delta x_j$.

When the equilibrium moves from the *initial* to the *third* position, the value of the preference function will change by

$$\delta\varphi = \varphi_1 \delta x_1 + \varphi_2 \delta x_2 + \cdots + \varphi_n \delta x_n\,,$$

and when it shifts from the *initial* to the *second* position, the change will be

$$d\varphi = \varphi_1 dx_1 + \varphi_2 dx_2 + \cdots + \varphi_n dx_n\,.$$

Suppose now that $\delta\varphi$ and $d\varphi$ are equal, i.e.,

$$\varphi_1(\delta x_1 - dx_1) + \varphi_2(\delta x_2 - dx_2) + \cdots + \varphi_n(\delta x_n - dx_n) = 0\,.$$

This, together with (11), leads to

$$[\text{ii}] \qquad p_1(\delta x_1 - dx_1) + p_2(\delta x_2 - dx_2) + \cdots + p_n(\delta x_n - dx_n) = 0\,.$$

But from (9),

$$p_1(\delta x_1 - \delta\alpha_1) + p_2(\delta x_2 - \delta\alpha_2) + \cdots + p_n(\delta x_n - \delta\alpha_n) = \delta\alpha$$

(*third* equilibrium),

$$p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n + (x_k - \alpha_k) dp_k = 0$$

(*second* equilibrium).

Thus,

$$\delta\alpha + \sum_{s=1}^{n} p_s \delta\alpha_s = -(x_k - \alpha_k) dp_k\,.$$

This equation represents the total income effect. The change in the equilibrium quantity of X_j from the *initial* to the *third* position (i.e., the income effect) is given by

$$\delta x_j = \frac{\partial x_j}{\partial \alpha}\delta\alpha + \sum_{s=1}^{n} \frac{\partial x_j}{\partial \alpha_s}\delta\alpha_s = \frac{\partial x_j}{\partial \alpha}\delta\alpha + \sum_{s} p_s \frac{\partial x_j}{\partial \alpha}\delta\alpha_s\,,$$

using (16). Finally, we have

$$\delta x_j = \left\{\delta\alpha + \sum_s p_s\delta\alpha_s\right\}\frac{\partial x_j}{\partial\alpha} = -(x_k - \alpha_k)\frac{\partial x_j}{\partial\alpha}dp_k ,$$

which is nothing but the first term on the right hand side of [i].

3.3. *The Substitution Term and the Substitution Effect.* For the *fourth* equilibrium we consider infinitesimal changes in the initial holdings of money and all goods, $\delta\alpha, \delta\alpha_1, \cdots, \delta\alpha_n$, and a change dp_k in the price of X_k. The equilibrium values in the new position are represented by

$$x_1 + \delta x_1, x_2 + \delta x_2, \cdots, x_n + \delta x_n, \lambda + \delta\lambda .$$

The value of the preference function will change by

$$\delta\varphi = \varphi_1\delta x_1 + \varphi_2\delta x_2 + \cdots + \varphi_n\delta x_n$$

when the equilibrium moves from the *initial* to the *fourth* position. Substituting (11) into this last expression, we obtain

[iii] $$\delta\varphi = \lambda(p_1\delta x_1 + p_2\delta x_2 + \cdots + p_n\delta x_n) .$$

On the other hand, from (11), it follows that

[iv] $$-p_i\delta\lambda + \sum_{s=1}^{n}\varphi_{is}\delta x_s = \lambda\varepsilon_{ik}dp_k , \qquad i = 1, 2, \cdots, n .$$

Let the value of the preference function be unchanged when the equilibrium shifts from the *initial* to the *fourth* position, i.e., $\delta\varphi = 0$. Then, from [iii],

[v] $$p_1\delta x_1 + p_2\delta x_2 + \cdots + p_n\delta x_n = 0 .$$

From equations [iv] and [v] we easily find that

$$\delta x_j = \sigma_{jk}dp_k , \qquad j = 1, 2, \cdots, n .$$

This is the substitution effect of price change on X_j, which coincides with the second term on the right hand side of [i].

3.4. *Price Changes of Unclassified Goods: A Law of Proportionality.** In this section we examine the effect of price changes of the commodities of the first group, i.e., unclassified goods, on equilibrium quantities of commodities of the second group. Let the price of good X_i change while all other prices and initial holdings of all goods are kept unchanged. Then, it follows from (18) and (19) that

* Translator's note: Unclassified goods are those of the first group. They have not been classified as separable goods, but they may or may not be separable.

$$p_{l+1}\frac{\partial x_{l+1}}{\partial p_1} + p_{l+2}\frac{\partial x_{l+2}}{\partial p_1} + \cdots + p_n\frac{\partial x_n}{\partial p_1} = \frac{\partial \gamma}{\partial p_1}, \tag{29}$$

$$-p_{l+1}\frac{\partial \lambda}{\partial p_1} + \varphi_{l+r,1}\frac{\partial x_1}{\partial p_1} + \varphi_{l+r,2}\frac{\partial x_2}{\partial p_1} + \cdots + \varphi_{l+r,l}\frac{\partial x_l}{\partial p_1} + \varphi_{l+r,l+1}\frac{\partial x_{l+1}}{\partial p_1} + \cdots + \varphi_{l+r,n}\frac{\partial x_n}{\partial p_1} = 0, \quad r = 1, 2, \cdots, n-l. \tag{30}$$

From (19) and (22),

$$\varphi_{l+r,i} = \nu_i \varphi_{l+r} = \lambda \nu_i p_{l+r}.$$

Substituting into the first line of (30) above, we obtain

$$-p_{l+r}\left\{\frac{\partial \lambda}{\partial p_1} - \lambda \sum_{j=1}^{l} \nu_j \frac{\partial x_j}{\partial p_1}\right\} + \sum_{s=1}^{n-l} \varphi_{l+r,l+s}\frac{\partial x_{l+s}}{\partial p_1} = 0, \tag{31}$$

$$r = 1, 2, \cdots, n-l.$$

From a comparison of the simultaneous equation system (29) and (31) with that of (20) and (23) if follows that

$$\frac{\dfrac{\partial \lambda}{\partial p_1} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial p_1}}{\dfrac{\partial \gamma}{\partial p_1}} = \frac{\dfrac{\partial \lambda}{\partial \alpha} - \lambda \sum \nu_j \dfrac{\partial x_j}{\partial \alpha}}{\dfrac{\partial \lambda}{\partial \alpha}},$$

$$\frac{\dfrac{\partial x_{l+s}}{\partial p_1}}{\dfrac{\partial \gamma}{\partial p_1}} = \frac{\dfrac{\partial x_{l+s}}{\partial \alpha}}{\dfrac{\partial \gamma}{\partial \alpha}}, \qquad s = 1, 2, \cdots, n-l.$$

Reformulating, we obtain

$$\frac{\dfrac{\partial x_{l+1}}{\partial p_1}}{\dfrac{\partial x_{l+1}}{\partial \alpha}} = \frac{\dfrac{\partial x_{l+2}}{\partial p_1}}{\dfrac{\partial x_{l+2}}{\partial \alpha}} = \cdots = \frac{\dfrac{\partial x_n}{\partial p_1}}{\dfrac{\partial x_n}{\partial \alpha}}$$

$$= \frac{\dfrac{\partial \lambda}{\partial p_1} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial p_1}}{\dfrac{\partial \lambda}{\partial \alpha} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial \alpha}} = \frac{\dfrac{\partial \gamma}{\partial p_1}}{\dfrac{\partial \gamma}{\partial \alpha}}. \tag{32}$$

Adding $(x_1 - \alpha_1)$ to each of these ratios and considering (27) and (28), we get

$$(33)\quad \frac{\sigma_{i+1,1}}{\dfrac{\partial x_{i+1}}{\partial \alpha}} = \frac{\sigma_{i+2,1}}{\dfrac{\partial x_{i+2}}{\partial \alpha}} = \cdots = \frac{\sigma_{n,1}}{\dfrac{\partial x_n}{\partial \alpha}} = \frac{\sigma_{n1} - \lambda \sum_j \nu_j \sigma_{j1}}{\dfrac{\partial \lambda}{\partial \alpha} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial \alpha}} = \frac{\omega_{n1}}{\dfrac{\partial \gamma}{\partial \alpha}},$$

where

$$(34)\quad \omega_{n1} = \frac{\partial \gamma}{\partial p_1} + (x_1 - \alpha_1)\frac{\partial \gamma}{\partial \alpha}.$$

This last equation (34) is the substitution term in the expression for the rate of change in the total expenditure on the commodities of the second group in response to a change in the price of X_1.

It follows also from (32) and (33) that

$$(35)\quad \frac{\dfrac{\partial x_{i+1}}{\partial p_1}}{\sigma_{i+1,1}} = \frac{\dfrac{\partial x_{i+2}}{\partial p_1}}{\sigma_{i+2,1}} = \cdots = \frac{\dfrac{\partial x_n}{\partial p_1}}{\sigma_{n,1}} = \frac{\dfrac{\partial \lambda}{\partial p_1} - \lambda \sum_j \nu_j \dfrac{\partial x_j}{\partial p_1}}{\sigma_{n1} - \lambda \sum_j \nu_j \sigma_{j1}} = \frac{\dfrac{\partial \gamma}{\partial p_1}}{\omega_{n1}}.$$

These equations refer to changes in the price of X_1, but similar results also hold for any one of goods $X_2, \cdots, X_i$ of the first group. On the other hand, the propensity to consume a separable good is proportional to the "proper" propensity to consume associated with it. We therefore obtain the following law of proportionality of separable goods with respect to changes in the prices of unclassified goods: *The rate of change, as well as the substitution term, of the equilibrium quantity of a separable good in response to a change in the price of an unclassified good is proportional to the "proper" propensity to consume the separable good (at the point of common equilibrium).*

It must be noted that the ratios in (32), (33), and (35) are not necessarily positive. If the ratios in (35), for instance, are positive, the rate of change in the demand for each separable good will have the same sign as its substitution term, but if they are negative, the rate will have a different sign from that of the substitution term. If the ratios in (32) are positive, a rise in the price of an unclassified good will increase demand for superior goods and decrease demand for inferior goods. If the ratios are negative, the reverse is true.

3.5. *Price Changes of Separable Goods.* Let the price of one of the commodities of the second group, say X_n, change, while all other prices and the initial holdings of all goods remain unchanged. We examine the effect on the equilibrium quantities of the commodities of the second group.

Differentiating (18) and (19) with respect to p_n, we obtain

$$(36)\quad p_{i+1}\frac{\partial x_{i+1}}{\partial p_n}+p_{i+2}\frac{\partial x_{i+2}}{\partial p_n}+\cdots+p_n\frac{\partial x_n}{\partial p_n}=-(x_n-\alpha_n)+\frac{\partial\gamma}{\partial p_n},$$

$$-p_{i+1}\frac{\partial\lambda}{\partial p_n}+\varphi_{i+r,1}\frac{\partial x_1}{\partial p_n}+\varphi_{i+r,2}\frac{\partial x_2}{\partial p_n}+\cdots+\varphi_{i+r,i}\frac{\partial x_i}{\partial p_n}$$

$$+\varphi_{i+r,i+1}\frac{\partial x_{i+1}}{\partial p_n}+\cdots+\varphi_{i+r,n}\frac{\partial x_n}{\partial p_n}=\lambda\varepsilon_{i+r,n},$$

$$r=1,2,\cdots,n-l.$$

The latter equation can be rewritten [see (19) and (22)] as

$$(37)\quad -p_{i+r}\left\{\frac{\partial\lambda}{\partial p_n}-\lambda\sum_{j=1}^{i}\nu_j\frac{\partial x_j}{\partial p_n}\right\}+\sum_{s=1}^{n-l}\varphi_{i+r,i+s}\frac{\partial x_{i+s}}{\partial p_n}=\lambda\varepsilon_{i+r,n},$$

$$r=1,2,\cdots,n-l.$$

Putting

$$(38)\quad \frac{\partial\gamma}{\partial\alpha}\rho_{i+s,n}=\frac{\partial\gamma}{\partial\alpha}\cdot\frac{\partial x_{i+s}}{\partial p_n}+\left\{(x_n-\alpha_n)-\frac{\partial\gamma}{\partial p_n}\right\}\frac{\partial x_{i+s}}{\partial\alpha},$$

$$\frac{\partial\gamma}{\partial\alpha}\rho_{0n}=\frac{\partial\gamma}{\partial\alpha}\left\{\frac{\partial\lambda}{\partial p_n}-\lambda\sum_{j=1}^{i}\nu_j\frac{\partial x_j}{\partial p_n}\right\}$$

$$+\left\{(x_n-\alpha_n)-\frac{\partial\gamma}{\partial p_n}\right\}\left\{\frac{\partial\lambda}{\partial\alpha}-\lambda\sum_{j}\nu_j\frac{\partial x_j}{\partial\alpha}\right\},$$

we obtain the following equations from (20), (23), (36) and (37):

$$(39)\quad p_{i+1,n}+p_{i+2}\rho_{i+2,n}+\cdots+p_n\rho_{nn}=0,$$

$$-p_{i+r}\rho_{0n}+\varphi_{i+r,i+1}\rho_{i+1,n}+\varphi_{i+r,i+2}\rho_{i+2,n}+\cdots+\varphi_{i+r,n}\rho_{nn}=\lambda\varepsilon_{i+r,n},$$

$$r=1,2,\cdots,n-l.$$

Let the determinant of the coefficients be

$$D^{(i)}=\begin{vmatrix} 0 & p_{i+1} & \cdots\cdots & p_n \\ p_{i+1} & \varphi_{i+1,i+1} & \cdots\cdots & \varphi_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ p_n & \varphi_{n,i+1} & \cdots\cdots & \varphi_{nn} \end{vmatrix},$$

and the cofactor of the r-th column, s-th row element be $D^{(i)}_{r,s}$. Then,

$$D^{(i)}\rho_{i+s,n}=\lambda D^{(i)}_{n-i+1,s+1},\qquad -D^{(i)}\rho_{0n}=\lambda D^{(i)}_{n-i+1,1}.$$

From (38) we obtain

$$(40)\quad \frac{\partial x_{i+s}}{\partial p_n}=\left\{\frac{\partial\gamma}{\partial p_n}-(x_n-\alpha_n)\right\}\frac{\partial x_{i+s}}{\partial\alpha}\Big/\frac{\partial\gamma}{\partial\alpha}+\rho_{i+s,n},$$

$$s=1,2,\cdots,n-l,$$

which represents the effect of the price change on the equilibrium quantity of X_{l+s}.

If we consider the relationships

$$\frac{\partial \beta}{\partial \alpha} + \frac{\partial \gamma}{\partial \alpha} = 1 , \qquad \frac{\partial \beta}{\partial p_n} + \frac{\partial \gamma}{\partial p_n} = 0 ,$$

we can rewrite (40) in the following Slutsky form:

$$\frac{\partial x_{l+s}}{\partial p_n} = -(x_n - \alpha_n)\frac{\partial x_{l+s}}{\partial \alpha} + \sigma_{l+s,n} , \qquad s = 1, 2, \cdots, n-l ,$$

(41)

$$\sigma_{l+s,n} = \rho_{l+s,n} - \left\{ \frac{\partial \beta}{\partial p_n} + (x_n - \alpha_n)\frac{\partial \beta}{\partial \alpha} \right\} \frac{\partial x_{l+s}}{\partial \alpha} \Big/ \frac{\partial \gamma}{\partial \alpha} .$$

3.6. *Substitution and the "Proper" Preference Function.* In his analysis of the substitution effect Hicks keeps the value of the "general" preference function constant. In this section we define substitution in a slightly different way, i.e., where the value of the "proper" preference function is kept constant. We thus give a new definition to the substitution effect.

Consider a new equilibrium when the initial holdings of money and other goods are subject to infinitesimal changes $\delta\alpha$, $\delta\alpha_1, \cdots, \delta\alpha_n$, while the price of a separable good X_n alone undergoes an infinitesimal change dp_n. We call this equilibrium position the *fifth* equilibrium. The quantities of all commodities and the value of λ will be changed by $\delta x_1, \delta x_2, \cdots, \delta x_n$, and $\delta\lambda$ when the equilibrium shifts from the *first* to *fifth* position. The "proper" preference function η (as given in footnote 6) will be subject to a change

$$\begin{aligned} d\eta &= \varphi_{l+1}\delta x_{l+1} + \varphi_{l+2}\delta x_{l+2} + \cdots + \varphi_n \delta x_n \\ &= \lambda(p_{l+1}\delta x_{l+1} + p_{l+2}\delta x_{l+2} + \cdots + p_n \delta x_n) . \end{aligned}$$

If the shift in equilibrium is such as to keep the value of the "proper" preference function constant, then

(i) $$p_{l+1}\delta x_{l+1} + p_{l+2}\delta x_{l+2} + \cdots + p_n \delta x_n = 0 ,$$

while from (19) it follows that

$$-p_{l+r}\delta\lambda + \sum_{j=1}^{l} \varphi_{l+r,j}\delta x_j + \sum_{s=1}^{n-l} \varphi_{l+r,l+s}\delta x_{l+s} = \lambda \varepsilon_{l+r,n} dp_n ,$$

or, substituting (22) into the left hand side of the expression,

(ii) $$-p_{l+r}(\delta\lambda - \lambda \sum_j \nu_j \delta x_j) + \sum_s \varphi_{l+r,l+s}\delta x_{l+s} = \lambda \varepsilon_{l+r,n} dp_n ,$$

$$r = 1, 2, \cdots, n-l .$$

Comparing the simultaneous equations (i) and (ii) with (39), we get

$$\delta x_{l+s} = \rho_{l+s,n} dp_n , \qquad s = 1, 2, \cdots, n-l .$$

The right hand side of this equation is equivalent to the second term on the right hand side of equation (40), multiplied by dp_n. We may thus call δx_{l+s} and $\rho_{l+s,n}$ respectively the substitution *effect* (for good X_{l+s}) and the substitution *term* in terms of the "proper" preference function of the second-group commodities.

3.7. *The "Proper" Preference Function and the Income Effect.* Let us now consider an equilibrium when the price of a single separable good X_n changes infinitesimally by dp_n, while all other prices and all initial holdings remain unchanged; this we call the *sixth* equilibrium. We also consider the *seventh* equilibrium, which occurs when the initial holdings of money and all commodities are infinitesimally changed by $\delta\alpha, \delta\alpha_1, \cdots, \delta\alpha_n$, with all prices kept constant. The shifts of equilibrium from the *first* to the *sixth* and *seventh* positions will induce changes in the equilibrium quantities of commodities

$$dx_1, dx_2, \cdots, dx_n \qquad (\textit{sixth} \text{ equilibrium}) ;$$
$$\delta x_1, \delta x_2, \cdots, \delta x_n \qquad (\textit{seventh} \text{ equilibrium}) .$$

The value of the "proper" preference function will be changed by

$$d\eta = \varphi_{l+1} dx_{l+1} + \varphi_{l+2} dx_{l+2} + \cdots + \varphi_n dx_n \qquad (\textit{sixth} \text{ equilibrium}) ;$$
$$\delta\eta = \varphi_{l+1} \delta x_{l+1} + \varphi_{l+2} \delta x_{l+2} + \cdots + \varphi_n \delta x_n \qquad (\textit{seventh} \text{ equilibrium}) .$$

When the changes in the initial holdings of money and commodities, in moving from the *first* equilibrium to the *seventh*, are such as to make $d\eta = \delta\eta$, namely

(iii)
$$\varphi_{l+1}\delta x_{l+1} + \varphi_{l+2}\delta x_{l+2} + \cdots + \varphi_n \delta x_n = \varphi_{l+1} dx_{l+1} + \varphi_{l+2} dx_{l+2} + \cdots + \varphi_n dx_n ,$$

we call the change δx_j in the equilibrium quantity of good X_j the income effect for X_j (in terms of the "proper" preference function). The income effect for X_{l+s} is

(iv)
$$\delta x_{l+s} = \frac{\partial x_{l+s}}{\partial \alpha}\delta\alpha + \frac{\partial x_{l+s}}{\partial \alpha_1}\delta\alpha_1 + \cdots + \frac{\partial x_{l+s}}{\partial \alpha_n}\delta\alpha_n$$
$$= \frac{\partial x_{l+s}}{\partial \alpha}(\delta\alpha + p_1\delta\alpha_1 + p_2\delta\alpha_2 + \cdots + p_n\delta\alpha_n) ,$$

using (16). On the other hand, from

$$p_1(x_1 - \alpha_1) + p_2(x_2 - \alpha_2) + \cdots + p_n(x_n - \alpha_n) = \alpha ,$$

we obtain

$$\text{(v)} \qquad \sum_{j=1}^{l} p_j dx_j + \sum_{s=1}^{n-l} p_{l+s} dx_{l+s} + (x_n - \alpha_n) dp_n = 0 ,$$

and

$$\text{(vi)} \qquad \sum_{j=1}^{l} p_j (\delta x_j - \delta \alpha_j) + \sum_{s=1}^{n-l} p_{l+s} (\delta x_{l+s} - \delta \alpha_{l+s}) = \delta \alpha .$$

(v), together with (iii), leads to

$$\text{(vii)} \qquad \sum_{j=1}^{l} p_j \delta x_j = \delta \alpha + p_1 \delta \alpha_1 + \cdots + p_n \delta \alpha_n - \sum_{s=1}^{n-l} p_{l+s} dx_{l+s} .$$

Since

$$\sum_{j=1}^{l} p_j \delta x_j = \sum_{j=1}^{l} p_j \left(\frac{\partial x_j}{\partial \alpha} \delta \alpha + \frac{\partial x_j}{\partial \alpha_1} \delta \alpha_1 + \cdots + \frac{\partial x_j}{\partial \alpha_n} \delta \alpha_n \right)$$

$$= \sum_{j=1}^{l} p_j \frac{\partial x_j}{\partial \alpha} (\delta \alpha + p_1 \delta \alpha_1 + \cdots + p_n \delta \alpha_n)$$

as a consequence of (16), we obtain

$$\sum_{j=1}^{l} p_j \delta x_j = \frac{\partial \beta}{\partial \alpha} (\delta \alpha + p_1 \delta \alpha_1 + p_2 \delta \alpha_2 + \cdots + p_n \delta \alpha_n) .$$

This, together with (vii), gives us

$$\left(1 - \frac{\partial \beta}{\partial \alpha} \right) (\delta \alpha + p_1 \delta \alpha_1 + \cdots + p_n \delta \alpha_n) - \sum_{s=1}^{n-l} p_{l+s} dx_{l+s} = 0 .$$

We have, therefore,

$$\text{(viii)} \qquad \delta \alpha + p_1 \delta \alpha_1 + \cdots + p_n \delta \alpha = \frac{\sum_s p_{l+s} dx_{l+s}}{\dfrac{\partial \gamma}{\partial \alpha}} .$$

Now

$$\sum_{s=1}^{n-l} p_{l+s} dx_{l+s} = -(x_n - \alpha_n) dp_n - \sum_{j=1}^{l} p_j dx_j$$

because of (v).

$$\sum_{s=1}^{n-l} p_{l+s} dx_{l+s} = -(x_n - \alpha_n) dp_n - \sum_j p_j \frac{\partial x_j}{\partial p_n} dp_n$$

$$= -\left\{ \frac{\partial \beta}{\partial p_n} + (x_n - \alpha_n) \right\} dp_n$$

$$= \left\{\frac{\partial\gamma}{\partial p_n} - (x_n - \alpha_n)\right\}dp_n , \qquad \because \frac{\partial\beta}{\partial p_n} + \frac{\partial\gamma}{\partial p_n} = 0 .$$

Substituting into (viii), we obtain

$$\delta\alpha + p_1\delta\alpha_1 + \cdots + p_n\delta\alpha_n = \frac{\left\{\frac{\partial\gamma}{\partial p_n} - (x_n - \alpha_n)\right\}dp_n}{\frac{\partial\gamma}{\partial\alpha}} ,$$

which, together with (iv), produces

$$\delta x_{1+s} = \frac{\left\{\frac{\partial\gamma}{\partial p_n} - (x_n - \alpha_n)\right\}\frac{\partial x_{1+s}}{\partial\alpha}dp_n}{\frac{\partial\gamma}{\partial\alpha}} .$$

This is equivalent to the first term on the right hand side of equation (40) multiplied by dp_n. We consequently call it the income term in terms of the "proper" preference function of the commodities of the second group.

The rate of change $\partial x_{1+s}/\partial p_n$ in the equilibrium quantity of X_{1+s}, a commodity of the second group, is equal to the sum of the Hicksian income and substitution terms if expressed in the Slutsky form. If it is expressed in the form of (40), it is equal to the sum of the income and substitution terms as used in this paper. In other words, the difference between $\sigma_{1+s,n}$ and $\rho_{1+s,n}$ is a part of the substitution term if we consider substitution of commodities in terms of the "general" preference function for all commodities. If we consider substitution in terms of the "proper" preference function for the separable commodities of the second group this difference is a part of the income term. Furthermore, the next section will show that this difference is equivalent to the difference between the substitution terms before and after the commodities of the first group are excluded from the preference field, and that this difference may make the substitution terms take different signs in the two cases. (See Section 3.9 below.)

3.8. *General Changes versus Proper Changes.* The commodities of the second group are separable from the commodities of the first group. Therefore, as shown in Section 2.4, all or some of the commodities of the first group may be excluded from the preference field without altering the equilibrium quantity of each commodity of the second group if money income is properly adjusted.

In order to compare the substitution terms before and after all goods

of the first group are excluded from the preference field, let us define the "proper" preference function of the commodities of the second group as in footnote 6. Equate money income to m, and exclude all first-group commodities from the preference field. Then, we obtain by (15) and (24) the following equations:

$$p_{l+1}\left[\frac{\partial x_{l+1}}{\partial p_n}\right] + p_{l+2}\left[\frac{\partial x_{l+2}}{\partial p_n}\right] + \cdots p_n\left[\frac{\partial x_n}{\partial p_n}\right] = -(x_n - \alpha_n)\,,$$

$$-p_{l+r}\left[\frac{\partial \mu}{\partial p_n}\right] + \sum_{s=1}^{n-l} \varphi_{l+r,l+s}\left[\frac{\partial x_{l+s}}{\partial p_n}\right] = \lambda \varepsilon_{l+r,n}\,, \qquad r = 1, \cdots, n-l\,.$$

Combining these with (25), we obtain

$$\begin{aligned} &p_{l+1}[\sigma_{l+1,n}] + p_{l+2}[\sigma_{l+2,n}] + \cdots + p_n[\sigma_{nn}] = 0\,, \\ &-p_{l+r}[\sigma_{0n}] + \sum_{s=1}^{n-l} \varphi_{l+r,l+s}[\sigma_{l+s,n}] = \lambda \varepsilon_{l+s,n}\,, \quad r = 1, 2, \cdots, n-l\,, \end{aligned} \tag{42}$$

where

$$[\sigma_{0n}] = \left[\frac{\partial \mu}{\partial p_n}\right] + (x_n - \alpha_n)\left[\frac{\partial \mu}{\partial m}\right],$$

and

$$[\sigma_{l+s,n}] = \left[\frac{\partial x_{l+s}}{\partial p_n}\right] + (x_n - \alpha_n)\left[\frac{\partial x_{l+s}}{\partial m}\right].$$

The brackets denote the exclusion of the first-group commodities.

We know that the determinant of the coefficients of (42) is not equal to zero if the stability conditions are satisfied. Therefore, referring to (39), we readily find that

$$\rho_{l+s,n} = [\sigma_{l+s,n}]\,, \qquad s = 1, 2, \cdots, n-l\,. \tag{43}$$

Thus, *the substitution term in terms of the "proper" preference function of the separable commodities is equivalent to the "proper" substitution term (at the point of common equilibrium)*. This result applies not only to changes in price of X_n but also to those of any other separable good. It must also be noted that it holds irrespective of the number of unclassfied goods. In view of these properties, we call this relation *the principle of invariance of the substitution effect* in terms of the "proper" preference function.

(40) and (41) may be rewritten [cf. (26) and (43)] as

$$\frac{\partial x_{l+s}}{\partial p_n} = \left[\frac{\partial x_{l+s}}{\partial p_n}\right] + \frac{\partial \gamma}{\partial p_n}\left[\frac{\partial x_{l+s}}{\partial m}\right] = \left[\frac{\partial x_{l+s}}{\partial p_n}\right] - \frac{\partial \beta}{\partial p_n}\left[\frac{\partial x_{l+s}}{\partial m}\right],$$

$$\sigma_{i+s,n} = [\sigma_{i+s,n}] - \left\{\frac{\partial\beta}{\partial p_n} + (x_n - \alpha_n)\frac{\partial\beta}{\partial\alpha}\right\}\left[\frac{\partial x_{i+s}}{\partial m}\right].$$

From this it follows that

$$\frac{\frac{\partial x_{i+1}}{\partial p_n} - \left[\frac{\partial x_{i+1}}{\partial p_n}\right]}{\left[\frac{\partial x_{i+1}}{\partial m}\right]} = \frac{\frac{\partial x_{i+2}}{\partial p_n} - \left[\frac{\partial x_{i+2}}{\partial p_n}\right]}{\left[\frac{\partial x_{i+2}}{\partial m}\right]} = \cdots = \frac{\frac{\partial x_n}{\partial p_n} - \left[\frac{\partial x_n}{\partial p_n}\right]}{\left[\frac{\partial x_n}{\partial m}\right]}$$

$$= \frac{\partial\gamma}{\partial p_n}$$

and

$$\frac{\sigma_{i+1,n} - [\sigma_{i+1,n}]}{\left[\frac{\partial x_{i+1}}{\partial m}\right]} = \frac{\sigma_{i+2,n} - [\sigma_{i+2,n}]}{\left[\frac{\partial x_{i+2}}{\partial m}\right]} = \cdots = \frac{\sigma_{nn} - [\sigma_{nn}]}{\left[\frac{\partial x_n}{\partial m}\right]}$$

$$= -\frac{\partial\beta}{\partial p_n} - (x_n - \alpha_n)\frac{\partial\beta}{\partial\alpha},$$

where $p_1(x_1 - \alpha_1) + p_2(x_2 - \alpha_2) + \cdots + p_i(x_i - \alpha_i) = \beta$.

We now have a *law of proportionality* relating changes in the price of a separable good and equilibrium quantities of the separable goods: *The differences between the "general" and "proper" rates of change in the equilibrium quantities of separable goods are proportional to the respective "proper" propensities to consume them (at the point of common equilibrium). The same statement holds for differences between the "general" and "proper" substitution terms.*[8]

3.9. *Signs of the "General" and "Proper" Substitution Terms: An Example.* Let

$$\eta = gk + hl, \text{ and } \varphi = f + \eta,$$

where we have the functions

$$f = x_1^a; \quad g = x_2^b; \quad h = x_3^c; \quad k = x_4^\beta; \quad \text{and} \quad l = x_4^\gamma;$$
$$(x_1, x_2, x_3, x_4 \geqq 0).$$

Assume φ to be a preference function of commodities X_1, X_2, X_3, and X_4. We have

$$\varphi_2 = \frac{bgk}{x_2}; \quad \varphi_3 = \frac{chl}{x_3}; \quad \varphi_4 = \frac{gk + \gamma hl}{x_4};$$

[8] A similar observation can be made within the second group if some commodities of the second group are separable from other commodities of the same group.

$$\varphi_{22} = \frac{b(b-1)gk}{x_2^2}\ ;\quad \varphi_{23} = 0\ ;\quad \varphi_{24} = \frac{b\beta gk}{x_2x_4}\ ;$$

$$\varphi_{33} = \frac{c(c-1)hl}{x_3^2}\ ;\quad \varphi_{34} = \frac{c\gamma hl}{x_3x_4}\ ;\quad \varphi_{44} = \frac{\beta(\beta-1)gk + \gamma(\gamma-1)hl}{x_4^2}\ ;$$

$$\begin{vmatrix} \varphi_{22} & \varphi_{24} \\ \varphi_{42} & \varphi_{44} \end{vmatrix} = \frac{bgk\{(1-b-\beta)\beta gk + (b-1)\gamma(\gamma-1)hl\}}{x_2^2x_4^2}\ ;$$

$$\begin{vmatrix} \varphi_{33} & \varphi_{34} \\ \varphi_{43} & \varphi_{44} \end{vmatrix} = \frac{chl\{(c-1)\beta(\beta-1)gk + (1-c-\gamma)\gamma hl\}}{x_3^2x_4^2}\ .$$

If a, b, c, β, and γ are all positive, and $a < 1$, $b + \beta < 1$, $c + \gamma < 1$, then both f and η, and hence φ, are subject to the principle of diminishing marginal value. The other commodities are separable from X_1 in φ; whereas any two of the remaining X's (X_2, X_3, and X_4) are not separable from the remaining one in η.

To ascertain the signs of the "general" and "proper" substitution terms, define

$$D = \begin{vmatrix} 0 & p_1 & \cdots & p_n \\ p_1 & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ p_n & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix} \quad \text{and} \quad D^{(3)} = \begin{vmatrix} 0 & p_3 & \cdots & p_n \\ p_3 & \varphi_{33} & \cdots & \varphi_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ p_n & \varphi_{n3} & \cdots & \varphi_{nn} \end{vmatrix}.$$

Then

$$D_{34} = \varphi_{11}D_{43}^{(3)} + p_1^2\varphi_{22}\varphi_{34}\ ;\quad D_{33} = \varphi_{11}D_{42}^{(3)} + p_1^2\varphi_{23}\varphi_{24}\ ; \tag{44}$$

$$\begin{aligned} D_{43}^{(3)} &= p_3p_4\varphi_{22} + p_2^2\varphi_{34} - p_2p_3\varphi_{24}\ ; \\ D_{42}^{(3)} &= p_2p_4\varphi_{23} + p_2^2\varphi_{34} - p_2p_3\varphi_{34}\ . \end{aligned} \tag{45}$$

In addition,

$$p_3D_{43}^{(3)} + p_2D_{42}^{(3)} = p_4(p_3^2\varphi_{22} + p_2^2\varphi_{23}) < 0\ .$$

Hence, either $D_{43}^{(3)}$ or $D_{42}^{(3)}$ must be negative. Multiplying both sides of (44) by λ^2, we obtain

$$\begin{aligned} \lambda^2D_{34} &= \varphi_{11}\left\{\lambda^2D_{43}^{(3)} + \frac{\varphi_1^2}{\varphi_{11}}(\varphi_{22}\varphi_{34})\right\}\ , \\ \lambda^2D_{33} &= \varphi_{11}\left\{\lambda^2D_{42}^{(3)} + \frac{\varphi_1^2}{\varphi_{11}}(\varphi_{22}\varphi_{24})\right\}\ . \end{aligned} \tag{46}$$

Since φ is subject to the principle of diminishing marginal value, the second term in the braces is positive. Therefore, if $D_{43}^{(3)}$ (or $D_{42}^{(3)}$) is positive, D_{34} (or D_{33}) is negative. But if $D_{43}^{(3)}$ (or $D_{42}^{(3)}$) is negative, D_{34} (or D_{33}) may be positive or negative depending on the size of the first

factor φ_1^2/φ_{11} of the second term in braces. The first term and the second factor ($\varphi_{22}\varphi_{34}$ or $\varphi_{23}\varphi_{24}$) of the second term in the braces are, however, independent of x_1, while the first factor of the second term is

$$\frac{\varphi_1^2}{\varphi_{11}} = \frac{\{f'(x_1)\}^2}{f''(x_1)} ,$$

i.e., a function of x_1 alone. Moreover, either $D_{43}^{(3)}$ or $D_{42}^{(3)}$ must be negative. Therefore, if $D_{42}^{(3)}$, for example, is negative, we can make D_{33} negative by proper selection of the function $f(x_1)$, i.e., X_1. On the other hand, the substitution terms of the other commodities with respect to a change in the price of X_4 are

$$\sigma_{43} = \frac{\lambda D_{34}}{D} ; \qquad \sigma_{42} = \frac{\lambda D_{32}}{D} ; \qquad \text{("general" substitution terms)}$$

$$[\sigma_{43}] = \frac{\mu D_{43}^{(3)}}{D^{(3)}} ; \quad [\sigma_{42}] = \frac{\mu D_{42}^{(3)}}{D^{(3)}} ; \qquad \text{("proper" substitution terms)}$$

while $D > 0$ and $D^{(3)} < 0$.

Thus, when $D_{42}^{(2)}$ and D_{32} are both negative, the "proper" substitution term $[\sigma_{42}]$ is positive and the "general" substitution term σ_{42} is negative. (The "proper" preference function as defined in footnote 6 will equate λ and μ at the equilibrium point. We shall, therefore, write both of them as λ henceforth.) In short, commodity X_1 is independent of other commodities and, if chosen properly, may cause the "proper" substitution term (of X_2 or X_3, with respect to a change in the price of X_4) to have the same or opposite sign as the "general" substitution term. A numerical illustration will clarify this point.

Let $b = 2/3$; $c = 1/3$; $\beta = 1/4$; and $\gamma = 1/2$, so that

$$\eta = x_2^{2/3}x_4^{1/4} + x_3^{1/3}x_4^{1/3} ,$$

and let $p_2 = 1.28$; $p_3 = 0.96$; and $p_4 = 1.53$. Let the initial money stock be 66.72 and holdings of all other commodities zero. The equilibrium quantities of X_2, X_3 and X_4 (X_1 being excluded) with respect to the "proper" preference function η are given by

$$x_2 = 27; \quad x_3 = 8; \quad x_4 = 16; \quad \text{and} \quad \lambda = \frac{100}{288} = \frac{5^2}{2^5 3^2} .$$

Our determinants take the following corresponding values:

$$\lambda^2 D_{43}^{(2)} = \frac{bcghkl}{x_2^2 x_3 x_4}[(b\gamma - \beta)gk - (1-b)\gamma hl] = \frac{1}{2^3 3^7} ;$$

$$\lambda^3 D_{43}^{(3)} = \frac{bcghkl}{x_2x_3^2x_4}[(c\beta - \gamma)hl - (1-c)\beta gk] = -\frac{19}{2^6 3^4};$$

$$\varphi_{23}\varphi_{34} = \frac{bcghkl}{x_2^2x_3x_4}(b-1)\gamma = -\frac{1}{2^6 3^7};$$

$$\varphi_{33}\varphi_{24} = \frac{bcghkl}{x_2x_3x_4^2}(c-1)\beta = -\frac{1}{2^6 3^4}.$$

Hence, from (46),

$$D_{34} = \frac{1}{2^6 3^7 \lambda^3}(\varphi_{11} - \varphi_1^2) = \frac{2^3}{3^3 5^4}(\varphi_{11} - \varphi_1^2),$$

(47)

$$D_{33} = -\frac{1}{2^6 3^4 \lambda^3}(38\varphi_{11} + \varphi_1^2) = -\frac{1}{5^4}(38\varphi_{11} + \varphi_1^2).$$

Thus,

$$D_{43}^{(3)} > 0; \quad D_{34} < 0; \quad D_{43}^{(3)} < 0.$$

Hence,

$$[\sigma_{43}] < 0; \quad \sigma_{43} < 0; \quad [\sigma_{43}] > 0.$$

The sign of D_{33}, however, (and that of σ_{43}) is indeterminate.

Take two variants of the preference function f of the commodity X_1

(i) $$f = 6x_1^{1/3},$$

(ii) $$f = 2x_1^{3/4},$$

and let $p_1 = 2.16$, all other prices remaining unchanged. Money income is increased to 101.28. When X_1 is included in the preference field, there will be no change in the equilibrium quantities of X_2, X_3, and X_4, and the equilibrium quantity of X_1 will be 16, whether function (i) or (ii) is taken.[9] For, in both cases,

$$f'(16) = \frac{3}{4}; \quad \frac{f'(16)}{p_1} = \frac{5^3}{2^3 3^3} \quad (= \lambda \text{ previously});$$

$$101.28 - p_1x_1 = 101.28 - 34.56 = 66.72 \quad (= \text{money income previously}).$$

To determine the sign of the "general" substitution term σ_{43}, we find

Case (i): From (47),

$$D_{33} = -\frac{\varphi_{11}}{5^4}\left(38 + \frac{\varphi_1^2}{\varphi_{11}}\right) = -\frac{f''}{5^4}\left(38 + \frac{f'^2}{f''}\right)$$

[9] When X_1 is added to the preference field, we can take $f(16) + \eta$ for the "proper" preference function of X_2, X_3, and X_4; this corresponds to that given in footnote 6. The elements of the determinant D are unaffected by this choice.

$$= \frac{1}{5^4} \cdot \frac{3}{2} \cdot \frac{1}{2^5}(38 - 24) = \frac{3 \cdot 7}{2^6 \cdot 5^4}; \qquad \therefore \ \sigma_{42} > 0;$$

Case (ii):

$$D_{42} = \frac{1}{5^4} \cdot \frac{3}{8} \cdot \frac{1}{2^5}(38 - 48) = -\frac{3}{2^7 \cdot 5^3}; \qquad \therefore \ \sigma_{42} < 0 .$$

That is, the "general" substitution term σ_{42} has the same sign as the "proper" substitution term in case (i) but has the opposite sign in case (ii). In Hicksian terms, case (ii) implies that X_2 and X_4 are substitutes in a group of three goods X_2, X_3 and X_4 but become complements if another good X_1 is added to this group. It must be noted that X_1 is independent of the other three goods. This fact leads to some doubts about the validity of Hicks' definitions.

4. INDEPENDENT GOODS

4.1. *An Exact Definition of Independent Goods.* Assume that X_2, X_3, $\cdots$, X_n are separable from X_1. Then, from Section 2.1, we know that

$$\varphi_{j1} = \nu_1 \varphi_j , \qquad j = 2, 3, \cdots, n .$$

Therefore,

$$\frac{\partial}{\partial x_1}\left(\frac{\varphi_j}{\varphi_1}\right) = \frac{1}{\varphi_1^2}(\varphi_{j1}\varphi_1 - \varphi_{11}\varphi_j) = \frac{1}{\varphi_1^2}(\nu_1\varphi_1 - \varphi_{11})\varphi_j ,$$

and

$$\frac{\varphi_1}{\varphi_j} \cdot \frac{\partial}{\partial x_1}\left(\frac{\varphi_j}{\varphi_1}\right) = \frac{1}{\varphi_1}(\nu_1\varphi_1 - \varphi_{11}) .$$

Setting the right hand side equal to θ, we obtain

$$(48) \qquad \frac{\frac{\partial}{\partial x_1}\left(\frac{\varphi_2}{\varphi_1}\right)}{\frac{\varphi_2}{\varphi_1}} = \frac{\frac{\partial}{\partial x_1}\left(\frac{\varphi_3}{\varphi_1}\right)}{\frac{\varphi_3}{\varphi_1}} = \cdots = \frac{\frac{\partial}{\partial x_1}\left(\frac{\varphi_n}{\varphi_1}\right)}{\frac{\varphi_n}{\varphi_1}} = \theta ,$$

which is obviously invariant with the choice of preference function. We call θ the percentage rate of change in the substitution rate with respect to X_1. In general, θ is a function of $x_1, x_2, \cdots, x_n$. In the special case where θ is independent of $x_2, x_3, \cdots, x_n$, i.e., θ is a function of x_1 alone or constant, the preference function φ will depend on the sum of a function of x_1 alone and another function of $x_2, x_3, \cdots, x_n$. This proposition will be proved below.

It follows from (48) that

$$\log\frac{\varphi_j}{\varphi_1} = \int\theta dx_1 + \chi_j\,, \qquad j = 2, 3, \cdots, n\,,$$

where the χs are functions of $x_2, x_3, \cdots, x_n$. Then,

$$\frac{\varphi_j}{\varphi_1} = \frac{\eta_j(x_2, \cdots, x_n)}{f_1(x_1)}\,, \tag{49}$$

where

$$f_1 = \exp\left(-\int\theta dx_1\right) \qquad \text{and} \qquad \eta_j = \exp(\chi_j)\,.$$

Let η_{jk} denote the partial derivative of η_j with respect to $x_k (k \geqq 2)$

$$\eta_{jk} = \frac{f_1}{\varphi_1^2}(\varphi_1\varphi_{jk} - \varphi_j\varphi_{1k}) = \frac{f_1}{\varphi_1^2}(\varphi_1\varphi_{jk} - \nu_1\varphi_j\varphi_k)\,.$$

Hence,

$$\eta_{jk} = \eta_{kj}\,, \qquad \left.\begin{matrix} j \\ k \end{matrix}\right\} = 2, 3, \cdots, n\,,$$

so that

$$f_1dx_1 + \eta_2dx_2 + \cdots + \eta_ndx_n$$

is the total differential. Integrating, we obtain

$$\begin{aligned}\int(f_1dx_1 + \eta_2dx_2 + \cdots + \eta_ndx_n) &= \int f_1dx_1 + \int(\eta_2dx_2 + \cdots + \eta_ndx_n) \\ &= f(x_1) + \eta(x_2, \cdots, x_n)\,.\end{aligned}$$

On the other hand, from (49) it follows that

$$\begin{aligned}d\varphi &= \varphi_1dx_1 + \varphi_2dx_2 + \cdots + \varphi_ndx_n \\ &= \frac{\varphi_1}{f_1}(f_1dx_1 + \eta_2dx_2 + \cdots + \eta_ndx_n)\,.\end{aligned}$$

Therefore, φ is a function of $(f + \eta)$, i.e.,

$$\varphi = F(f + \eta)\,. \tag{50}$$

Conversely, when the preference function φ has the form of (50), the equations (48) will hold, and their value θ will be a function of x_1 alone or a constant.

We define a commodity X_1 to be independent of other commodities when the percentage rates of change in the rates of substitution of other commodities for X_1 (with respect to X_1) are all equal to each other and independent of the values of $X_2, X_3, \cdots, X_n$, i.e., when equations (48) hold and θ is independent of $x_2, x_3, \cdots, x_n$. This defini-

tion is based on the way in which commodities are chosen by the individual, that is, on properties of the preference function. The fact that X_1 is independent of other goods does not, however, imply that X_1 is unrelated to other goods in use. In the special case where each good is independent of all the others, the definition reduces to Allen's[10] and should be regarded as a revision of Pareto's definition.[11]

In the previous discussion all the marginal degrees of preference were treated as positive. η_{ij} was also positive because of (49). The derivative F' in (50), therefore, was positive. Thus, when X_1 is independent of other goods, $f + \eta$ can be used as the preference function. In other words we choose a φ which satisfies the conditions

$$\varphi_{1j} = \varphi_{j1} = 0 , \qquad j = 2, 3, \cdots, n .$$

In this case,

$$\theta = -\frac{\varphi_{11}}{\varphi_1} .$$

5. CHANGES IN THE PRICES OF INDEPENDENT GOODS

5.1. *Changes in the Expenditure on an Independent Good.* Let X_1 be independent of other commodites $X_2, X_3, \cdots, X_n$. For the preference function φ, take one which satisfies the following conditions:

$$\varphi_{1j} = \varphi_{j1} = 0 , \qquad j = 2, 3, \cdots, n .$$

Let D and $D^{(1)}$ be the determinants given in Section 3.9. Define

$$\Delta = \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix} \quad \text{and} \quad \Delta^{(1)} = \begin{vmatrix} \varphi_{22} & \cdots & \varphi_{2n} \\ \cdots & \cdots & \cdots \\ \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} .$$

Set

$$p_1(x_1 - \alpha_1) = \beta , \quad \beta + \gamma = \alpha ,$$
$$p_2(x_2 - \alpha_2) + p_3(x_3 - \alpha_3) + \cdots + p_n(x_n - \alpha_n) = \gamma .$$

We find

$$\frac{\partial \beta}{\partial p_1} = (x_1 - \alpha_1) + p_1 \frac{\partial x_1}{\partial p_1} = (x_1 - \alpha_1) + p_1 \left\{ -(x_1 - \alpha_1) \frac{\partial x_1}{\partial \alpha} + \frac{\lambda D^{(1)}}{D} \right\} ,$$

while

[10] R. G. D. Allen, "A Reconsideration of the Theory of Value, Part II," *Economica*, XIV (1934), 214.

[11] Vilfredo Pareto, "Économie Mathématique," *Encyclopédie des sciences mathématique*, (1911), 612. *Manuel d'économie politique*, (Paris: Giard, 1927), 2nd ed. 576.

$$D\frac{\partial x_1}{\partial \alpha} = -p_1 \Delta^{(1)}, \quad D = \varphi_{11} D^{(1)} - p_1^2 \Delta^{(1)}, \quad \varphi_1 = \lambda p_1 .$$

Hence,

$$\frac{\partial \beta}{\partial p_1} = \left\{ \frac{(x_1 - \alpha_1)\varphi_{11}}{\varphi_1} + 1 \right\} \frac{\varphi_1 D^{(1)}}{D} .$$

Now set

$$\varepsilon = -\frac{(x_1 - \alpha_1)\varphi_{11}}{\varphi_1} \left[= (x_1 - \alpha_1)\theta \right] .$$

See Section 4.1. We now obtain a formula for the rate of change in expenditure on X_1

$$\frac{\partial}{\partial p_1}\{p_1(x_1 - \alpha_1)\} = -(\varepsilon - 1)\frac{\varphi_1 D^{(1)}}{D} , \tag{51}$$

where $\varphi_1 D^{(1)}/D$ is negative because of the stability conditions and invariant with the choice of the preference function φ. ε is also invariant with the choice of the preference function (because of invariance of θ). For the preference function in general, $\Phi = F(\varphi)$,

$$\varepsilon = -\frac{(x_1 - \alpha_1)\Phi_j}{\Phi_1} \frac{\partial}{\partial x_1}\left(\frac{\Phi_1}{\Phi_j}\right), \qquad j = 2, 3, \cdots, n . \tag{52}$$

We may call the right hand side of (52) the elasticity of the marginal rate of substitution of X_1 for X_j. When X_1 is independent of other commodities, the elasticity of X_1 for X_j is the same for all X_j; this is clear from (52). In terms of this definition, (51) can be stated as follows: *When good X_1 is independent of other goods, the rate of change in expenditure on it in response to a change in its own price is positive, zero, or negative, according as the elasticity of its marginal rate of substitution is greater than, equal to, or less than unity.*[13]

[13] The rate of change in expenditure has the following relationship to the rate of change in λ:

$$\frac{\partial \beta}{\partial p_1} = \frac{D^{(1)}}{\Delta^{(1)}} \frac{\partial \lambda}{\partial p_1} ,$$

which can be rewritten as

$$\frac{\partial \beta}{\partial p_1} = \left(\frac{D}{\Delta} + \frac{p_1^2}{\varphi_{11}}\right) \frac{\partial \lambda}{\partial p_1} ,$$

since

$$D = \varphi_{11} D^{(1)} - p_1^2 \Delta^{(1)}, \quad \Delta = \varphi_{11} \Delta^{(1)} .$$

The formula corresponds to that derived by Pareto for the case of any good independent of other goods. (See Pareto, *Manuel*, p. 581 and *Économie mathématique*, p. 629; also Henry Schultz, "Interrelations of Demand, Price and Income," *Journal of Political Economy*, XLIII (1935), p. 451). Pareto's formula, therefore, holds not only when every

5.2. *Changes in the Equilibrium Quantities of Separable Goods.* On the basis of the same assumptions we employed in Section 5.1, we obtain

$$\frac{\partial \gamma}{\partial p_1} = -\frac{\partial \beta}{\partial p_1} = (\varepsilon - 1)\frac{\varphi_1 D^{(1)}}{D} .$$

From (26),

$$\frac{\partial x_j}{\partial \alpha} = \frac{\partial \gamma}{\partial \alpha}\left[\frac{\partial x_j}{\partial m}\right], \qquad j = 2, 3, \cdots, n ,$$

where the factor in brackets represents the "proper" propensity to consume X_j when X_1 is excluded from the preference field. Combining these two relations with (32),

$$\frac{\partial x_j}{\partial p_1} = \frac{\partial x_j}{\partial \alpha} \cdot \frac{\partial \gamma}{\partial p_1} \Big/ \frac{\partial \gamma}{\partial \alpha}, \qquad j = 2, 3, \cdots, n ,$$

we obtain

$$\frac{\partial x_j}{\partial p_1} = (\varepsilon - 1)\frac{\varphi_1 D^{(1)}}{D}\left[\frac{\partial x_j}{\partial m}\right], \qquad j = 2, 3, \cdots, n , \tag{53}$$

where the "proper" propensity to consume is measured at the point of common equilibrium. Equation (53) is a formula measuring the effect of a change in the price of an independent good X_1 on the equilibrium quantities of separable goods. Since the second factor on the right hand side is negative, the sign of this effect is determined by the magnitude of the elasticity ε relative to unity and by the sign of the "proper" propensity to consume.

On the basis of (51) and (53), we can tabulate the effect of a rise in the price of X_1, an independent good, on the equilibrium quantities (demanded) of other goods.

$\varepsilon - 1$	Demand for X_1	Expenditure on X_1	Demand for a "properly" superior good*	Demand for a "properly" inferior good*
−	Decreased	Decreased	Increased	Decreased
0	Decreased	Unchanged	Unchanged	Unchanged
+	Decreased	Increased	Decreased	Increased

* Superior or inferior goods are called "properly" superior or inferior goods when X_1 is excluded from the preference field.

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good is independent of other goods, but also when a particular good is independent of other goods. The formulas (51) and (53) differ from Pareto's in that ours make it evident that the elasticity of the rate of substitution is invariant with the preference function chosen; they differ also in their clarification of the significance of the factors which determine the rates of change of equilibrium quantities in terms both of this elasticity and of the "proper" propensity to consume.

From the view point of common sense, a rise in the price of good A will decrease demand for itself and lower the value of the preference function. The greater this decrease in the value of the preference function, the more the individual will try to limit the decrease in his demand for good A in his attempt to keep the value of the preference function stable. This attempt may lead to an increase in expenditure on good A and a decrease in expenditure on the other goods. On the other hand, when the value of the preference function is only slightly reduced, the individual may tend to lessen the decrease in the value of the preference function by adjusting his demands for other commodities rather than for good A, according to how strongly he wants to have these commodities. Formulas (52) and (53) demonstrate this relationship clearly.

5.3. *The Substitution Term.* After a little calculation we find that the substitution term σ_{j1}, $(j \geqq 2)$, can be written as

$$\sigma_{j1} = \frac{\partial x_j}{\partial p_1} + (x_1 - \alpha_1)\frac{\partial x_j}{\partial \alpha} = \left\{(\varepsilon - 1)\frac{\varphi_1 D^{(1)}}{D} + (x_1 - \alpha_1)\frac{\partial \gamma}{\partial \alpha}\right\}\left[\frac{\partial x_j}{\partial m}\right].$$

Since

$$\varepsilon\varphi_1 = -(x_1 - \alpha_1)\varphi_{11}, \qquad D\frac{\partial \gamma}{\partial \alpha} = \varphi_{11}D^{(1)},$$

we have

$$\sigma_{j1} = -\frac{\varphi_1 D^{(1)}}{D}\left[\frac{\partial x_j}{\partial m}\right], \qquad j = 2, 3, \cdots, n. \tag{54}$$

The "general" substitution term has the same sign as the "proper" propensity to consume each good (except X_1) (at the point of common equilibrium).

From this result we may conclude that Hicks' definitions distinguishing complementarity from substitutability according to the sign of the substitution term are not applicable to independent goods. This conclusion, together with the example in Section 3.9, challenges the validity of his definitions.

From (53) and (54) we obtain

$$\frac{\partial x_j}{\partial p_1} = -(\varepsilon - 1)\sigma_{j1}, \qquad j = 2, 3, \cdots, n. \tag{55}$$

This formula represents the relationship between the substitution term and the effect of a change in the price of an independent good on the equilibrium quantities of other goods. It demonstrates that *the*

expression for the effect of such a change has the same or opposite sign as the substitution term according as the elasticity ε is greater or less than unity.

In summary, we find that the formulas (53), (54), and (55) give the proportionality factors in our laws of proportionality (see Section 3.4) when the price of an independent good changes. It must be particularly noted that the signs of the proportionality factors are determined exclusively by the size of the elasticity ε relative to unity; the elasticity itself is determined solely by the equilibrium quantity of the independent good and is independent of the equilibrium quantities of the other goods.

Kyoto University, Japan

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A HISTORICAL NOTE ON PROFESSOR SONO'S THEORY OF SEPARABILITY

By Michio Morishima

Masazo Sono, a famous mathematician in Japan, first became interested in economics through his friendship with Yasuma Takata, who has been called the "Japanese Marshall" by M. Bronfenbrenner [1]. Like Slutsky's celebrated article, Sono's work on the separability of goods [15] published in Japanese in the midst of World War II has remained unknown for a long time without attracting attention from English-speaking economists. Four years after the publication of Sono's paper, however, Lenontief [5, 6] discussed the same problem completely independently, and recent contributions to the subject have been made by Hicks [3], Strotz [16, 17], Gorman [2], Miss Rajaoja [11] and others. In view of the importance of Sono's results, we have felt it desirable to reproduce his paper in translation for those economists who cannot read Japanese.

Strotz [17] and Miss Rajaoja [11] gave two definitions of a separable utility function, which they called the weak and the strong definition. According to this classification, Sono's work amounts to having exhausted the properties of weak separability. In fact, Leontief's Theorem I in [5] (or Proposition I in [6]) is implied by Sono's Propositions I and II in [15, (section 2)]; Strotz's equations (9), and (14), and footnote (12) in [16] are nothing but Sono's "laws of proportionality" in sections 3.4, 2.5 and 3.8 respectively.

Leontief's Theorem II in [5] (or Proposition IV in [6]) regarding an *additively* separable function is based on the definition of strong separability; no equivalent of the theorem is found in Sono's paper [15]. It is evident, however, that if there are but two subsets of goods into which the whole set is separated, then strong separabilily is reduced to weak separability. The theorem is inapplicable to the case where the whole set of goods is partitioned in *two* additively separable subsets. On the other hand, Sono's discussion of independent goods (in section 4.1, 5.1–5.3) acquaints us with the conditions and the properties of additive separability in the two-subset case. In this respect, Leontief and Sono are complements of each other.

The Sono-Leontief theory is set forth in abstract terms and may be connected with various interpretations, two possible examples of which have been suggested by Leontief [6]. First, the internal

* Manuscript received February 13, 1961.

structure of a production function can effectively be analyzed by separating the overall production function into a number of intermediate production functions. For example, the overall production function of steel which connects the final output of finished steel with the original inputs (i.e., the quantities of various kinds of labor, tools, and materials used in mining coal and iron ore as well as those used directly in the operation and maintenance of blast furnaces) can be broken down by explicitly introducing the intermediate products, coal and ore, into three simpler functions, each involving fewer variables. Second, we may assume, as a first approximation, that for any two goods in a given budget category, the marginal rate of substitution between them is independent of the demand for any goods not in that category, e.g., the marginal rate of substitution between evening dress and kimono should be independent of the consumption of bread and rice. This assumption of weak separability was used by Strotz [16] and Miss Rajaoja [11]. If it is sharpened to additive separability or homogeneous separability[1], it implies that consumers first allocate expenditure among the several budget categories and then decide how best to spend each budget allotment on the commodities within the category, with no further reference to purchases in other categories. See Gorman [2], Strotz [17] and Rajaoja [11].

For the purpose of studying intrinsically related goods, Hicks [3 (chapter 17)] suggested a new approach. He thinks of the consumer as choosing between certain objectives, and then deciding between alternative means of reaching those objectives. The relative valuations of those goods which are means of reaching the same objective are determined by their relative contributions to that objective, and are independent of the remaining goods which are devoted to other objectives. The relation between the means and the objective is of a technical character, being similar to the relation between factor inputs and commodity outputs. We may assume that the functions which give us relations between means and objectives are homogeneous of degree one[2]. Thus Hicks' idea can be translated into the language of homogeneous separability. See Morishima [8]. The consumer will decide, more or less as the entrepreneur decides, between alternative means. The Hicks theory may, therefore, be considered a synthesis of traditional demand theory and theory of the firm, which have

[1] If each "proper" utility function (in the sense of Sono) is homogeneous of degree one, a utility function is said to be homogeneously separable. Cf. Gorman [2].

[2] That is, constant returns to scale prevail between means and objectives.

hitherto been separated.

Another application of the Sono-Leontief theory was made by Yasui [18] and Morishima [7]. They make the assumption that monetary goods (money and bonds) and consumption goods are separable from one another; that is to say, the consumer's utility function is assumed to be a function of the proper utility function of consumption goods and the proper utility (or liquidity) function of monetary goods. This assumption, together with others, enables us to establish the following laws of demand: (1) Suppose that a consumer's inital assets are positive, and that the marginal propensity to hold cash and bonds (with respect to an increase in the initial assets) is less than unity. Then the demand for proper superior consumption goods decreases and the demand for proper inferior goods increases when all prices rise proportionally. (2) If a rise in the interest rate increases a consumer's savings, it will decrease his demand for proper superior consumption goods and increase his demand for proper inferior consumption goods and the number of bonds held by him at the end of the period, and so on. (See [7].) These laws are based on Sono's "laws of proportionality."

Finally, it has been shown by Gorman [2], Nataf [9], and Okamoto and Morishima [10] that the separability assumption is basic to Klein's aggregation problem [4]. Hence it is also basic to the existence of an index of the quantity of capital—a problem recently raised by Mrs. Robinson [12, 13]. This last fact has been observed by Solow [14].

Osaka University, Japan.

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THE STRUCTURE OF MULTI-SECTOR DYNAMIC MODELS*

BY DALE W. JORGENSON

IN THIS PAPER we consider multi-sector generalizations of various Keynesian type macroeconomic models and investigate the existence, uniqueness, and nonnegativity of solutions to these models. On the basis of the seemingly elementary consideration that national income and capital stock or sectoral income and stock levels must be nonnegative, doubt is cast on the general validity of the well-known models of the trade cycle proposed by Samuelson [40, 41], Hicks [24], and Smithies [42], and the augmented dynamic input-output system of Leontief [33]. Using the same elementary consideration, we can deduce stringent conditions which must be satisfied by the ordinary dynamic input-output system, Hicks' "basic equation," and Duesenberry's multi-sector model of growth and fluctuations [11].

On the constructive side, techniques are provided for the analysis of solutions to the linear and piece-wise linear systems which dominate the literature on Keynesian macroeconomics. If the solutions to a macroeconomic model can be shown to be nonnegative for any nonnegative initial conditions, methods associated with the spectral theory of nonnegative matrices may be employed to characterize the solutions in detail. In particular, the existence and relative stability of a nonnegative characteristic solution are assured. Using more specific information, we may find it possible to obtain bounds to the equilibrium rate of growth.

Application of these techniques requires information only on the signs of elements in the matrices associated with the reduced form of a macroeconomic model. In many of the cases examined in this paper, this information can be obtained directly from the economic interpretation of the corresponding coefficients of investment and consumption functions or other relationships.

1. TWO THEOREMS ON NONNEGATIVITY

The investigation will be based on the following theorems:

THEOREM 1. *For a system of linear differential equations*

$$\dot{x} = Ax + f(t)\,, \tag{1.1}$$

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where A is a constant matrix, $f(t) \geqq 0$ for $t \geqq 0$ is a vector-valued function of t, $x(0) \geq 0$ is a vector of initial conditions, a necessary and sufficient condition that $x(t) \geq 0$ for $t \geqq 0$ is that there is a nonnegative scalar c so that $cI + A \geqq 0$.[1]

The corresponding result for difference equations is as follows:

THEOREM 2. *For a system of linear difference equations*

$$x(t) = Ax(t-1) + f(t), \tag{1.2}$$

where A is a constant matrix, $f(t) \geqq 0$ for $t \geqq 0$ is a vector-valued function of t, $x(0) \geq 0$ is a vector of initial conditions, a necessary and sufficient condition that $x(t) \geq 0$ for $t \geqq 0$ is that $A \geq 0$ and A has at least one element strictly positive in each column.[2]

These theorems provide simple criteria for the nonnegativity of solutions to macroeconomic models formulated as linear difference or differential equations or for piece-wise linear systems.[3] Proof of the theorems is based on a series of lemmas which provide alternative and equivalent conditions for the nonnegativity of solutions to systems of difference and differential equations in terms of certain functions of a matrix.

[1] That is, A is a Metzler matrix; off-diagonal elements are nonnegative. A recent review of mathematical properties of Metzler and related matrices together with a summary of economic applications of such matrices is contained in [34]. Although the following discussion is confined to models of the Keynesian type, some results are already available for models of the Walrasian type [33, 39]. In certain elementary cases, like the Leontief static system discussed below in Section 2, the difference between Walrasian and Keynesian models is simply a matter of the economic interpretation.

For vectors (and matrices) the following notation will be useful. If x is a vector with elements x_i, $x > 0$ if $x_i > 0$ for all i; $x \geqq 0$ if $x_i \geqq 0$ for all i; $x = 0$ if $x_i = 0$ for all i; $x \geq 0$ if $x \geqq 0$ and $x \neq 0$.

[2] This condition is equivalent to assumption (**) (i) of Kemeny, Morgenstern, and Thompson [29, (118)]. The system is also seen to satisfy assumption (**) (ii) since B, the output matrix, is the identity matrix, I. I am indebted to Professor Morishima for these observations.

[3] A system is piece-wise linear if its solutions may be represented as a continuous sequence of solutions to linear systems. Since the existence and uniqueness of solutions to each "phase" are usually assured, problems of existence center on the transition from one phase to another, that is, on the consistency of "switching" conditions [33, (72-76)]. For other examples of piece-wise linear systems, see [15, 16, 17]. So far as existence and uniqueness of solutions to piece-wise linear systems is concerned, it may be possible to approximate such equations by nonlinear equations which have continuously differentiable solutions and to apply standard techniques for analysis of such nonlinear equations [25, 36]. Here, however, few results are available for equations of order greater than two. The fact that the theory of positive matrices plays an important role in nonnegativity of solutions to linear equations suggests that it may be possible to make some progress in the nonlinear case through application of the theory of positive operators [32, 28].

LEMMA 1. *$Ax \geq 0$ for $x \geq 0$ if and only if $A \geq 0$ and A has at least one element strictly positive in each column.*

LEMMA 2. *A necessary and sufficient condition that $A^t \geq 0$ and A^t has at least one element strictly positive in each column for any integer t, nonnegative, is that $A \geq 0$ and A has at least one element strictly positive in each column.*

PROOF: Sufficiency is obtained by observing that the lemma is trivial for $t = 1$; suppose the lemma true for $t = n - 1$, then we may write

$$A^n = A \cdot A^{n-1},$$

where $A, A^{n-1} \geq 0$, and each matrix has at least one element strictly positive in each column. Applying Lemma 1 to each column of A^n considered as a vector, we observe that each such column is nonnegative and not zero. Necessity is obtained from consideration of $t = 1$.

Theorem 2 is an immediate consequence of Lemmas 1 and 2.

LEMMA 3.[4] *A necessary and sufficient condition that $e^{At} \geq 0$ for $t \geqq 0$ is that there is a positive scalar c so that $cI + A \geqq 0$.*

PROOF: Expanding the matrix exponential function e^{At} in power series

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots,$$

we observe that for small t, e^{At} has at least one negative element if there is no positive scalar c so that $cI + A \geqq 0$. For sufficiency we observe that the existence of such a positive scalar implies the nonnegativity of e^{At} for small t, but for any t and any n we have

$$(e^{At/n})^n = e^{At}.$$

Choose n sufficiently large so that $e^{At/n}$ is nonnegative; then, since e^{At} is nonsingular for any t [4, (166)], $e^{At/n}$ has at least one element strictly positive in each column. Hence, $e^{At} \geq 0$ by Lemma 2.

PROOF OF THEOREM 1. The general solution to the system (1.1) is

[4] Lemma 3, with a proof suggested by Karlin, and application of the results to differential equations are given in a slightly different form by Bellman [4, (172)]. Arrow [3, (6 and 14)] gives a different proof for the sufficiency part of Theorem 1. The matrix exponential function is discussed in detail by Bellman.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds\ .$$

If there is a nonnegative scalar c so that $cI + A \geqq 0$, $e^{At} \geq 0$ by Lemma 3. Since e^{At} is nonsingular, $x(0) \geq 0$ implies $e^{At}x(0) \geq 0$ by Lemma 1. Secondly, for $s \leqq t$, $e^{A(t-s)}f(s) \geqq 0$ for $f(s) \geqq 0$, by Lemma 1. Hence $\int_0^t e^{A(t-s)}f(s)ds \geqq 0$ for $t \geqq 0$. But $x(t)$ is the sum of two nonnegative vectors, one of which is not zero. Hence $x(t) \geq 0$ for $t \geqq 0$, which completes proof of sufficiency.

For necessity, let $f(t) = 0$ for all t; then $e^{At} \geq 0$ only if there is a nonnegative scalar c so that $cI + A \geqq 0$ by Lemma 3. By Lemma 1, $e^{At}x(0) \geq 0$ for $x(0) \geq 0$ only if $e^{At} \geq 0$ for all $t \geqq 0$, $x(t) = e^{At}x(0)$, which completes the proof.

Theorems 1 and 2 may be applied to piece-wise linear systems by making use of the observation that the initial conditions for each linear "phase" are the end conditions for the previous phase, so that the homogeneous part of such solutions can be written as the product of a sequence of matrix exponential or matrix power functions, each such function corresponding to one phase of the system.

In the following sections, Theorems 1 and 2 will be applied to a series of disaggregated macroeconomic models of the Keynesian type. All the models examined are based on a separation of total output into two parts—consumption and investment. Consumption is determined by some version of the consumption function; investment may be taken as autonomous, as in the Kahn dynamic multiplier [27], or as determined by the acceleration principle; i.e., investment is proportional to changes in income. The three versions of the acceleration principle to be examined are (i) unlagged or instantaneous adjustment of capital stock to changes in income, (ii) lagged adjustment with a fixed lag, and (iii) adjustment distributed over time (distributed lags). When combined with the consumption function, these alternative versions of the acceleration principle generate (i) the Harrod-Domar growth model [20, 21; 8, 9], (ii) the Samuelson-Hicks model of the trade cycle [40, 41; 24], and (iii) the Smithies-Duesenberry model of fluctuations and growth [42; 11]. The corresponding multi-sector models are (i) the dynamic input-output system of Hawkins and Leontief [22; 33], (ii) a multi-sector model of the trade cycle described by Allen [1, (364)], (iii) Duesenberry's multi-sector model [11, (222–25)].

2. MULTIPLIER THEORY

The simplest Keynesian macrodynamic theory is based on a deter-

mination of investment outside the model, i.e., investment is autonomous. In the discussion that follows it will be assumed that autonomous expenditures are nonnegative.

It is well known that disaggregation of the static multiplier of Keynes for determination of national income leads to a model which is formally identical with the Leontief static system of the open type [5; 18, 19; 44]

$$(I - A)x = y\,, \tag{2.1}$$

where x is a vector of sector income levels, $y \geq 0$ is a vector of final or autonomous demands, and $A \geq 0$ is an input-output matrix.[5] The conditions under which there is a unique, nonnegative solution to this system of equations have been carefully investigated by Solow [44] and by Hawkins and Simon [23]. Solow's results are as follows:

There exists a unique solution $x \geq 0$ of the system (2.1) for any $y \geq 0$ if (i) A is an irreducible input-output matrix with at least one row sum $\sum_j a_{ij}$ strictly less than one,[6] or (ii) A is a reducible input-output matrix, but arranged in normal form so that each irreducible submatrix along the main diagonal has at least one row sum strictly less than one [44, (36–38)].[7] In the disaggregated form of Kahn's dynamic multiplier [27], $x(t)$ and $y(t)$ are vectors with elements representing the income and autonomous demand for the output of the corresponding sector. If A is a matrix with nonnegative elements $[a_{ij}]$, where a_{ij} is the marginal (and average) propensity of the jth sector to consume the output of the ith sector, the dynamic multiplier may be represented in the form given by Goodwin and

[5] An input-output matrix is a matrix $A \geq 0$ with elements (a_{ij}) such that $\sum_j a_{ij} \leq 1$.

[6] A matrix is irreducible if it is not reducible. For a reducible matrix there is a simultaneous permutation of rows and columns such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11}, A_{22} are square submatrices. A matrix is decomposable if there is a simultaneous permutation of rows and columns such that

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

where A_{11}, A_{22} are square submatrices. Many different systems of terminology are used to describe reducible and decomposable matrices. For example, Solow [44] and Debreu and Herstein [7] use the terms decomposable and completely decomposable. The terminology employed here appears to be standard in the mathematical literature [13].

[7] The normal form of a reducible matrix A is

Solow [18, 19; 44]

$$x(t) = Ax(t-1) + y(t)\,. \tag{2.2}$$

Applying Theorem 2, we find that the solution

$$x(t) = A^t x(0) + y(t) + Ay(t-1) + \cdots + A^{t-1}y(1) \tag{2.3}$$

is nonnegative and not zero for any initial levels of income $x(0)$, nonnegative and not zero, if and only if the matrix $A \geq 0$ has at least one element strictly positive in each column. This latter condition means that each sector consumes the output of at least one other sector (possibly its own). If the coefficients a_{ij} are interpreted as technologically given constants, this assumption corresponds to Koopmans' "impossibility of the Land of Cockaigne" [30, (49–52); 31, (77–79)]. There is no *necessary* connection between the existence and uniqueness of a nonnegative solution to the multi-sector dynamic multiplier (2.2) and the convergence of the multiplier process to a stationary nonnegative solution

$$x = (I - A)^{-1}y\,, \tag{2.4}$$

where $y \geq 0$ is a constant vector. The *sufficient* condition that this solution of the model be stable and nonnegative given by Solow [44] also implies that the solution to the dynamic multiplier (2.2) is nonnegative for all values of time. The irreducibility of A implies that the complete solution (2.3) is nonnegative and *not zero* for all values of time.

3. THE DYNAMIC INPUT-OUTPUT SYSTEM

In the Harrod-Domar growth model, investment is taken as determined within the system by an unlagged accelerator.. In the discussion that follows the closed, continuous time formulation of the Harrod-Domar theory will be used. Consumption is determined by a static multiplier; investment is proportional to the rate of change of income.

In the dynamic input-output system A is an input-output matrix,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & A_{mm} \end{bmatrix},$$

where $A_{11}, A_{22}, \ldots, A_{mm}$ are irreducible square submatrices along the main diagonal. Such a normal form is unique except for the order of submatrices along the main diagonal [13, (89–92)].

and $B \geq 0$ is a stock-flow matrix with elements $[b_{ij}]$, showing the stock of the ith commodity required for production of one unit of the jth. The vectors $x(t)$ and $\dot{x}(t)$ have elements representing the output and rate of change of output of the corresponding sector. The complete dynamic input-output system may be written in the form

$$x(t) = Ax(t) + B\dot{x}(t) . \tag{3.1}$$

To simplify the following discussion, it will be assumed at the outset that B^{-1} exists and that $B^{-1}(I - A)$ is nonsingular and irreducible. The solution to the dynamic input-output system may be written

$$x(t) = e^{B^{-1}(I-A)t}x(0), \quad x(0) \geq 0 . \tag{3.2}$$

Various attempts have been made to obtain conditions that the solution of $x(t)$ be nonnegative and not zero [6, 14, 22, 26, 43]. Previous results may be summarized as follows: Since A is an input-output matrix, $(I - A)^{-1}$ exists and is nonnegative; hence $(I - A)^{-1}B$ is the product of two nonnegative matrices and is itself nonnegative. Since $B^{-1}(I - A)$ is irreducible, $(I - A)^{-1}B$ is an irreducible nonnegative matrix. By Frobenius' theorem [13, (65)], $(I - A)^{-1}B$ has a unique positive characteristic vector, say ζ, and no other characteristic vector which is nonnegative. Secondly, associated with this vector is a characteristic value of $(I - A)^{-1}B$, say $1/\gamma$, which is simple, positive, and largest in modulus of all the characteristic values of $(I - A)^{-1}B$. The corresponding uniquely nonnegative characteristic solution of (3.2), $e^{\gamma t}\zeta$, is relatively stable if and only if all characteristic values of $B^{-1}(I - A)$ are less than γ in real part [26]. Various other conditions on $B^{-1}(I - A)$ for relative stability have been given for two and three sector models [22, 14] using determinants; however, these conditions are equivalent to the condition on characteristic values just given. It will be shown that relative stability is necessary for nonnegativity of solutions to (3.1). However, no sufficient conditions for nonnegativity have been given in the literature.

To obtain a necessary and sufficient condition for nonnegativity of solutions to the dynamic input-output system, we apply Theorem 1. For nonnegativity of solutions to (3.1), it is necessary and sufficient that for the matrix $B^{-1}(I - A)$ there is a scalar c such that

$$cI + B^{-1}(I - A) \geqq 0 ,$$

that is, $B^{-1}(I - A)$ has only nonnegative off-diagonal elements. If $B^{-1}(I - A)$ is acyclic and irreducible, all diagonal elements of $B^{-1}(I - A)$

must be negative; otherwise, $B^{-1}(I - A)$ could not be the inverse of a positive matrix.[8] It can be shown that the condition for nonnegativity of solutions for the dynamic input-output system is sufficient for relative stability of the characteristic solution $e^{\gamma t}\zeta > 0$. Since the matrix $cI + B^{-1}(I - A)$ is nonnegative and irreducible, it has a uniquely nonnegative characteristic vector $\zeta > 0$ associated with a characteristic value $c + r > 0$, which is the characteristic value of $B^{-1}(I - A)$ largest in modulus and hence largest in real part. Then γ is the characteristic value of $B^{-1}(I - A)$ largest in real part, and $e^{\gamma t}\zeta$ is relatively stable. It should be observed that this sufficient condition for relative stability of the uniquely nonnegative solution makes no use of determinantal conditions or conditions on the characteristic values of the matrix $B^{-1}(I - A)$.[9]

4. THE LAGGED ACCELERATOR

In the complete theory of the trade cycle proposed by Hicks, Samuelson's linear multiplier-accelerator is combined with two nonlinear relations—a "floor" to capital decumulation and a "ceiling" to the rise of output imposed by capacity. In this section the linear part of the Samuelson-Hicks model, Hicks' elementary case, will be analyzed. In Section 6 a nonlinear model in which capital accumulation is irreversible will be discussed. For the elementary case, consumption is determind by the income of the previous period as in Kahn's dynamic multiplier; investment is determind by a lagged accelerator and is proportional to the change of income in the previous period.

[8] A matrix is acyclic if it is not cyclic. For a cyclic matrix, there is simultaneous permutation of rows and columns such that

$$A = \begin{bmatrix} 0 & A_1 & 0 \cdots 0 \\ 0 & 0 & A_2 \cdots 0 \\ \cdots & \cdots & \cdots \\ A_p & 0 & 0 \cdots 0 \end{bmatrix}.$$

The necessity of a condition involving acyclicity for the fact that a nonnegative irreducible matrix cannot have a nonnegative inverse was pointed out to me by Professor Morishima.

[9] A final observation on the dynamic input-output system is that if the system for the determination of output levels (4.3) has nonnegative solutions for all nonnegative initial conditions, the dual discussed by Morishima and Solow [34, 43] cannot have such solutions. The proof is as follows: If the output system has nonnegative solutions for any nonnegative initial conditions not zero, it is relatively stable. Then by the dual stability theorem [26], the dual is relatively unstable and therefore has solutions which have negative components for nonnegative initial conditions, not zero. Of course, this dual stability theorem does not hold for the open system.

It is possible to derive a multi-sector analogue to the Samuelson-Hicks model of the trade cycle along the lines suggested by Allen [1, (364)]. The marginal propensity to consume is replaced by an input-output matrix and the stock-flow coefficient by a stock-flow matrix. Let $x(t)$ be a vector with components equal to the level of output in each sector. The system may be written in the form

$$\begin{bmatrix} x(t) \\ x(t-1) \end{bmatrix} = \begin{bmatrix} A+B & -B \\ I & 0 \end{bmatrix} \begin{bmatrix} x(t-1) \\ x(t-2) \end{bmatrix} + \begin{bmatrix} y(t) \\ 0 \end{bmatrix}. \tag{4.1}$$

Applying Theorem 2, it is easily seen that the condition $B \geq 0$ is sufficient to imply that solutions to (4.1) cannot be nonnegative for all nonnegative initial conditions.

A number of alternative versions of the consumption function have been combined with the lagged accelerator to form a model of the trade cycle. Two examples will be discussed. Consumption is a fraction of the current period's income, $Ax(t)$, and consumption is some weighted average of the previous two periods' incomes, $A_1x(t-1) + A_2x(t-2)$. The resulting models are

$$\begin{aligned} x(t) &= (I-A)^{-1}Bx(t-1) - (I-A)^{-1}Bx(t-2) + (I-A)^{-1}y(t)\,, \\ x(t) &= (A_1+B)\,x(t-1) + (A_2-B)\,x(t-2) + y(t)\,. \end{aligned} \tag{4.2}$$

In the first case, if autonomous expenditure is nonnegative, the nonhomogeneous term is nonnegative, since for A an input-output matrix $(I-A)^{-1}$, if it exists, will be nonnegative. But $-(I-A)^{-1}B$ is nonpositive and not zero, so that the first model cannot satisfy the conditions for nonnegativity given by Theorem 2. In the second case, the conditions of Theorem 2 are satisfied if and only if $A_2 \geq B$, that is, if and only if the marginal propensity of each sector to consume from each other sector with respect to its own output, two periods previous, is greater than the corresponding stock-flow coefficient for the output of that sector. This condition appears to be economically implausible.

Essentially the same analysis can be applied to the various distributed lag consumption and investment models discussed by Hicks [24, (182 *et seq.*)] and by Allen [1, (228–39)]. It will suffice to consider Hicks' "basic equation," which may be written

$$x_t = \sum_{i}^{s} a_i x_{t-i} + \sum_{i}^{s-1} b_i (x_{t-i} - x_{t-i-1}) + y_t\,. \tag{4.3}$$

where the coefficients a_i, b_i are consumption and stock-flow coefficients,

x_t is the level of output in the current period and y_t is the level of autonomous expenditure. This system can be rewritten in matrix form as

$$(4.4)\qquad \begin{vmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{vmatrix} = \begin{vmatrix} a_1 + b_1 & a_2 + b_2 - b_1 & \cdots & a_p - b_{p-1} \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{vmatrix} + \begin{vmatrix} y_t \\ 0 \\ \vdots \\ 0 \end{vmatrix} .$$

Theorem 2 may be applied to demonstrate that the system has only nonnegative solutions if and only if

$$(4.5)\qquad \begin{aligned} &a_i + b_i \geqq b_{i-1}\,, \quad (i = 2, \cdots, p-1) \\ &a_p > b_{p-1}\,. \end{aligned}$$

The economic interpretation of the first set of conditions is that the marginal propensity to consume with respect to the income of the ith period must be greater than or equal to the difference between the accelerator coefficient associated with changes in output of the succeeding period and the coefficient associated with current changes in output. The second condition is simply that the final term in the series of consumption coefficients must be larger than the final term in the series of accelerator coefficients. If the effects of past income levels on consumption die out before the effects of such income levels on investment, these conditions would be violated.

If the conditions required for nonnegativity are satisfied, it may be observed that the matrix (4.4) is a nonnegative irreducible matrix; there is a unique, simple, positive characteristic root associated with a positive characteristic vector; furthermore, if $\sum_1^p a_i < 1$, the modulus of this characteristic value is less than one. There are no roots of the system with modulus greater than or equal to one, and the possibility of progressive equilibrium described by Hicks [24, (183)] is ruled out. The system (4.4) approaches a stationary state in essentially the same way as an ordinary dynamic multiplier like that of Solow [44] or the distributed lag multiplier described by Hicks [24, (180–82)]. A multi-sector analogue of the general distributed lag model is easy to formulate; the conditions required for nonnegativity of solutions are analogous to those of the second system (4.2).

5. THE CAPITAL STOCK ADJUSTMENT MODEL

The most significant difference between the older multiplier-acceler-

ator models and the models proposed by Smithies and Duesenberry is that in the latter investment is determined by income and capital stock. There are important analogies between the complete models considered by Smithies and Duesenberry [2, (46–50)]; however, a difference in the formulation of the consumption function turns out to be critical for the existence of nonnegative solutions. To begin the analysis, we consider Duesenberry's model, given by investment and consumption functions

$$\begin{aligned} I_t &= \alpha Y_{t-1} - \beta K_{t-1}\,; \quad \alpha, \beta > 0\,; \\ C_t &= a Y_{t-1} + b K_{t-1}\,; \quad a, b > 0\,; \end{aligned} \tag{5.1}$$

together with identities in income and capital stock

$$\begin{aligned} Y_t &= I_t + C_t\,; \\ K_t &= I_t + K_{t-1} - R_t\,; \end{aligned} \tag{5.2}$$

where I_t, C_t are investment and consumption, Y_t, K_t are income and capital stock, and R_t is physical depreciation, assumed to be a constant proportion of capital stock

$$R_t = kK_{t-1}\,; \quad 0 < k < 1\,. \tag{5.3}$$

Since investment and income are measured gross of depreciation, each must be nonnegative. It is easily seen that capital stock must also be nonnegative.

The model proposed by Smithies consists of two phases, each of which is linear; the complete model is piece-wise linear. The linear model for the first phase associated with expansion is analogous to that of Duesenberry.[10] To show this, we consider Smithies' investment and consumption functions with time trends omitted

$$\begin{aligned} I_t &= (\beta_1 + \beta_2 + \beta_3) Y_{t-1} - \beta_3 \sigma K_{t-1}\,; \quad \beta_1, \beta_2, \beta_3, \sigma > 0\,; \\ C_t &= (1 - \alpha_1 + \alpha_2) Y_t\,; \quad \alpha_1, \alpha_2 > 0\,. \end{aligned} \tag{5.4}$$

This system may be compared with equations (5.1); the principal difference is that capital stock does not enter the consumption function. The system is completed by two identities in income and capital

[10] To facilitate comparison with the Duesenberry model, Smithies' variable Y_F, full employment income, is transformed to K, capital stock, using the relation $Y_F = \sigma K$, where σ is Domar's coefficient [2, (47)]. In what follows, D_1 represents depreciation and D_2 extraordinary obsolescence [42, (21)]. The parameters δ_1 and δ_2 are the corresponding rates of depreciation, where depreciation is calculated by the declining balance method.

$$
\begin{aligned}
Y_t &= I_t + C_t\,, \\
K_t &= I_{t-1} + K_{t-1} - \frac{1}{\sigma}(D_1 + D_2)\,,
\end{aligned}
\tag{5.5}
$$

where $D_1 + D_2$ corresponds to depreciation. The second identity may be written

$$
K_t = I_{t-1} + K_{t-1} - (\delta_1 + \delta_2)K_{t-1} + \frac{\delta_2}{\sigma}\,Y_{t-1}\,. \tag{5.6}
$$

The complete system reduces to

$$
\begin{bmatrix} Y_t \\ K_t \end{bmatrix} = \begin{bmatrix} \dfrac{\beta_1 + \beta_2 + \beta_3}{\alpha_1 - \alpha_2} & \dfrac{-\beta_3\sigma}{\alpha_1 - \alpha_2} \\ \dfrac{\delta_2}{\sigma} + \alpha_1 - \alpha_2 & 1 - \delta_1 - \delta_2 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ K_{t-1} \end{bmatrix}. \tag{5.7}
$$

Applying Theorem 2, it is evident that since either the first or second coefficient in the first row of the matrix in (5.7) must be negative, Smithies' first phase model cannot satisfy the requirement that solutions be nonnegative for all nonnegative initial conditions.

Since the model is piece-wise linear there are implicit restrictions on the initial conditions. There remains the possibility that a given set of initial conditions would dictate a change of phase. Hence the second phase of the model must also be examined. The complete system becomes

$$
\begin{bmatrix} Y_t \\ K_t \end{bmatrix} = \begin{bmatrix} \dfrac{\beta_1}{\alpha_1} & \dfrac{-\beta_3\sigma}{\alpha_1} \\ \alpha_1 + \dfrac{\delta_2}{\sigma} & 1 - \delta_1 - \delta_2 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ K_{t-1} \end{bmatrix} + \begin{bmatrix} \dfrac{\alpha_2 + \beta_2 + \beta_3}{\alpha_1}\,\bar{Y} \\ -\alpha_2\,\bar{Y} \end{bmatrix}. \tag{5.8}
$$

Applying Theorem 2 as before, it is clear that the second phase does not satisfy the conditions required for nonnegativity; in particular, it is always possible to choose a level of excess capacity sufficiently high so that the actual level of income is negative in the succeeding period, applying either the model of phase one or that of phase two. The model which should be applied depends not on the initial conditions alone but on whether the initial value of income is the highest ever.

A multi-sector version of the capital stock adjustment model has been given by Duesenberry [11, (222–25)]. A single consumption function of essentially the same form as (5.1) is given by

$$
C_t = a\,Y_{t-1} + \sum_{1}^{n} b_i K_{i,\,t-1}\,, \tag{5.9}
$$

where $K_{i,\,t-1}$ is the capital stock in the ith sector. Individual investment functions are provided for all sectors:

$$(5.10)\qquad I_{i,\,t} = \alpha\, l_i\, Y_{t-1} - \beta_i K_{i,\,t-1}\,, \qquad (i = 1\,,\cdots,n)\,,\quad \sum_{i=1}^{n} l_i = 1\,,$$

where $I_{i,\,t}$ is gross investment in the ith industry and $l_i > 0$ is a constant of proportionality. Finally, there are two sets of identities

$$(5.11)\qquad \begin{aligned} Y_t &= C_t + \sum_{1}^{n} I_{i,\,t}\,,\\ K_{i,\,t} &= I_{i,\,t} + K_{i,\,t-1} - R_{i,\,t}\,, \qquad (i = 1\,,\cdots,n)\,, \end{aligned}$$

where $R_{i,\,t}$ is the physical depreciation in the ith industry, assumed proportional to the capital in that industry

$$(5.12)\qquad R_{i,\,t} = k_i K_{i,\,t-1}\,, \qquad (i = 1\,,\cdots,n)\,.$$

The complete system may be written in the form

$$(5.13)\qquad \begin{bmatrix} Y_t \\ K_{1t} \\ K_{2t} \\ \vdots \\ K_{nt} \end{bmatrix} = \begin{bmatrix} \alpha+a & b_1-\beta_1 & b_2-\beta_2 & \cdots & b_n-\beta_n \\ \alpha l_1 & 1-\beta_1-k_1 & 0 & \cdots & 0 \\ \alpha l_2 & 0 & 1-\beta_2-k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha l_n & 0 & 0 & \cdots & 1-\beta_n-k_n \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ K_{1,\,t-1} \\ K_{2,\,t-1} \\ \vdots \\ K_{n,\,t-1} \end{bmatrix}.$$

Necessary and sufficient conditions for nonnegativity of solutions to (5.13) are

$$(5.14)\qquad \begin{bmatrix} b_i \\ 1 \end{bmatrix} \geq \begin{bmatrix} \beta_i \\ \beta_i + k_i \end{bmatrix}, \qquad (i = 1\,,\cdots,n)\,.$$

If the conditions for nonnegativity (5.14) are satisfied, the matrix in (5.13) is nonnegative. If, in addition, the elements of the top row are positive, the matrix is irreducible. A nonnegative irreducible matrix has a uniquely positive characteristic vector associated with a positive, real, simple characteristic root largest in modulus of all the characteristic values of the matrix. The vector may be interpreted as the equilibrium proportions between the capital stocks held by each sector and the output of the economic system; the characteristic value is the long-run equilibrium growth factor. Clearly, the corresponding uniquely positive solution of the system is relatively stable so that, asymptotically, the movement of the economic system is characterized by balanced growth (or decline) at a constant rate of growth. The

rate of growth may be calculated by subtracting unity from the equilibrium growth factor.

6. IRREVERSIBILITY OF CAPITAL ACCUMULATION

Nonlinear relationships play an important role in extensions of the multiplier-accelerator models discussed in the previous sections. Specifically, irreversibility of capital accumlation has been discussed by Leontief [33], Hicks [24], Duesenberry [12], and Smithies [42]. In this section the existence of nonnegative solutions for the piece-wise linear model of Leontief will be investigated.

The possibility that the ordinary dynamic input-output system may generate negative output levels has led to many attempts to augment the ordinary system with certain additional relationships. In particular, Dorfman, Samuelson, and Solow [10, (335–45)] have interpreted the system as a model of optimal capital accumulation. An alternative proposal, made by Leontief [33, (68–76)], is to imbed the ordinary system in an augmented system in which capital accumulation is irreversible. As Leontief points out, the augmented system may be inconsistent. The problem of consistency has been discussed by Uzawa [45] and McManus [35]. It will be shown that for the augmented system there always exist economically meaningful initial conditions for which inconsistency will occur and that Leontief's proposal cannot surmount the difficulties associated with solutions having negative components for nonnegative initial conditions.

To begin the discussion, we follow Uzawa's lucid interpretation of the augmented dynamic input-output model. If all stocks are irreversible the model is given by the following system of differential equations:

$$(I - A)x = B^0 \max(\dot{x}, 0)\,, \tag{6.1}$$

where $B^0 \geqq 0$ is a matrix with elements b^0_{ij} such that

$$b^0_{ij} = \begin{cases} b_{ij} & \text{if } s_{ij} = b_{ij}x_j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6.2}$$

b_{ij} is an element of B, a stock-flow matrix; x_j is an element of the vector x of output levels; s_{ij} is the stock of the ith commodity held by the jth industry; and $\max(\dot{x}, 0)$ is a vector with elements, $\max(\dot{x}_j, 0)$. If the initial output and stock levels are denoted $x(0)$, $S(0)$, output must be less than capacity:

$$B[\operatorname{diag} x(0)] \leqq S(0)\,, \tag{6.3}$$

where diag $x(0)$ is a diagonal matrix with elements $x_j(0)$; furthermore, the usual nonnegativity conditions must be satisfied,

$$x(0) \geq 0 , \quad S(0) \geq 0 . \tag{6.4}$$

$S(0) \geq 0$ is implied by (6.3) and the condition that $x(0) \geq 0$ if at least one of the commodities corresponding to strictly positive elements of $x(0)$ cannot be produced without stocks. Any set of initial conditions $x(0)$, $S(0)$ satisfying (6.3) and (6.4) will be said to be *economically meaningful*.

A second set of relations, not given by Uzawa, completes the augmented dynamic input-output system

$$\dot{S} = B^0 \, (\text{diag } [\max (\dot{x}, 0)]) . \tag{6.5}$$

If all stocks are irreversible, stock levels are constant unless $s_{ij} = b_{ij}x_j$, and $\dot{x}_j > 0$.

For this model, we have the following result: For any augmented dynamic input-output system, if at least one off-diagonal element of A is not zero, there exist economically meaningful initial conditions $x(0)$, $S(0)$, for any B^0 such that there is no solution to the augmented dynamic input-output system.[11]

PROOF: Suppose $a_{ij} > 0$, $i \neq j$; then by Lemma 1, there is a vector $x^*(0) \geq 0$ so that $(I - A)\, x^*(0)$ has at least one negative element. For fixed B^0, choose $S^*(0)$ so that $s^*_{ij}(0) = b^0_{ij}\, x^*_j(0)$ if $b^0_{ij} = b_{ij}$, and $s^*_{ij}(0) > b^0_{ij}\, x^*_j(0)$ otherwise. Clearly, $x^*(0)$ and $S^*(0)$ comprise an economically meaningful set of initial conditions.

Since $B^0 \geq 0$ and $\max (\dot{x}, 0) \geqq 0$, $B^0 \max (\dot{x}, 0) \geqq 0$. Hence, for initial conditions $x^*(0)$, $S^*(0)$ the equality in (6.1) cannot hold.

The interpretation of this theorem is that for any of the possible "phases" of Leontief's dynamic input-output system there are economically meaningful initial conditions, nonnegative, for which no solution to the system exists. In Uzawa's formulation of the dynamic input-output system, it is assumed that all stocks are irreversible. For the general augmented dynamic input-output system, the relation (6.2) holds only for irreversible stocks. However, the usual causes to which the irreversiblility of capital accumulation is attributed—technical irreducibility as in the case of land or buildings or nontransferability as in the case of equipment and inventories—would imply that if a stock is irreversible in one sector, it must be irreversible in all others. It is easily observed that the proof just given goes through provided

[11] The form of the result and method of proof are due to Professor Morishima.

that there is just one commodity which is irreversible as a stock in every industry if the commodity is used as an input on current account in any industry except the industry which produces it. In such a case, there are economically meaningful initial conditions for which no solution to the system exists.

University of California (Berkeley), U.S.A.

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THE STRUCTURE OF MULTIPLIER-ACCELERATOR MODELS OF THE UNITED STATES ECONOMY, 1909-1951*

BY R. J. BALL AND EUGENE SMOLENSKY[1]

1. INTRODUCTION

IT IS a little over twenty years since Sir Roy Harrod [9], in a pioneering article, laid down the foundations of a great deal of theorizing about economic growth and cyclical fluctuations. Professor Samuelson [19] and Professor Domar [2] may also lay claim to be among the founders of this type of theory, the former through his recognition of the super-multiplier principle and the latter through his independent statement of the conditions for steady growth, which parallel those formulated by Harrod.

Since these initial impulses, developments have been pursued by Professor Hicks in the field of business cycles [10], Professor Smithies on the link between cycles and growth [20], Professor Leibenstein in the field of economic development [16] and many other contributors too numerous to mention.

In models of the Hicks-Smithies-Leibenstein type the numerical values of the structural parameters assume key importance. Professor Hicks, for example, accepts the hypothesis that the structure of his multiplier-accelerator equation is such as to generate explosive cycles bounded by the restraining influences of the "floor" and the "ceiling". Professor Smithies examines the behavior of his model under the assumption that the roots of the system are complex, leading to a steady cyclical path. The essential point is that the dynamic properties of the models turn on numbers that the theorist himself thinks "reasonable".

Since the theoretical models call for *ex ante* information, it is virtually impossible to derive, from empirical evidence, numerical

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parameters that can be identified with the algebraic coefficients of the theoretical models. In some cases the models postulated are thought to be only partial explanations of variations in the level of output and employment. Professor Smithies has carefully limited the application of his model to only a few of the economic fluctuations that have taken place in the United States over the last 50 years or so, and it is easy to sympathize with the view that there are several different kinds of fluctuations that may result from the particular behavior of the individual components of gross investment: Investment in plant and equipment, residential construction, and inventories, for each of which an alternative kind of model can be developed.

Yet, as best he can, the empirical worker and the policy maker must apply the theories at hand to interpret changes in the data available and make predictions as to the future course of events. This paper constitutes such an effort, an attempt to look into the results of applying some simple models of the multiplier-accelerator type to time series data for the United States over the period 1909-1951[2]. We have drawn on the body of existing theory in specifying the models considered, and the numerical estimates obtained may be suggestive, even if a one-to-one relationship with the theoretical models cannot be achieved.

The approach pursued below has evolved in response to several distinct objectives. First, we wanted to deal with simple models that, formally at least, bear a reasonable resemblance to theory. Multiplier-accelerator models have characteristically been stated in real terms, and the link between the real and monetary sectors, despite the work of Professor Hicks, has never been clearly established. All price effects and monetary variables have, therefore, been excluded from the analysis, and the relationships are postulated in real terms. The relationships have been made linear, which is also in keeping with this objective. Secondly, the aim has been to construct small models whose underlying logic is relatively clear and whose dynamic properties are fairly easy to examine. This does not necessarily imply that the models as a whole can be described as linear (e.g., Hicks' model is essentially nonlinear although his basic formulation is in linear terms). Indeed a significant nonlinearity is introduced in a rather crude way in the analysis below.

[2] We may record that this work had its roots in some discussion not of multiplier-accelerator models as such, but of economic growth inspired by Domar's observations [3] on an econometric model of economic growth. The growth aspect is by no means obliterated in what follows since in a naive way some allowance is made for shifts in productive technique and the growth of population.

Criteria for the "successful" application of forms of the multiplier-accelerator theories are not easy to come by, however. Initially, statistical criteria as to the goodness of fit, the reliability of the parameters, the character of the residual disturbances, and so on may be applied to the individual relationships. However the difficulty with these tests is that other simple models with alternative underlying hypotheses can probably be found that will in some sense "fit" the data about as well as the one which we have tried. We cannot test the conditional hypothesis on which the statistical analysis becomes based[3]. A further test that is intuitively appealing is that of prediction. The sample period employed in this study covers the years 1909–1951, and predictions have been made on a year-to-year basis for the period 1952–1957. Such a test is open to the same objection as the first. But it is at least suggestive. In pursuing economic analysis for the purpose of economic forecasting, we may at least hope to distinguish pragmatically between competing theories applied to aggregate data. A final relevant consideration is the kinds of dynamic behavior that the estimated structures imply. The sample period is one of considerable interest, for it includes a fairly "representative" collection of cyclical fluctuations imposed upon an overall trend of economic advancement. With regard to the possible patterns of dynamic behavior that models provide when estimated from this period, it becomes interesting to see whether the statistical parameters lead to explosive or damped behavior, each of which has its own set of assumptions in economic theory.

2. GENERAL FRAMEWORK

We may at the outset establish a general background or framework within which the equations of the models are specified. A distinction is drawn between the actual output of the economy at any moment of time and the capacity level of output, where output is the gross national product of the United States in constant dollars. Capacity output is defined to be the output that is attainable when all the factors of production are fully employed[4]. The maximum output of the economy is assumed to grow at a rate dependent on the rates of growth of the two basic factors of production, labor and capital, and shifts in productive technique due to technological change. The "state

[3] For some discussion of this problem see Theil [22].

[4] In our simple structures there is no distinction made between full employment and capacity output. This distinction may become significant in certain contexts, e.g., the analysis of price level changes. For a discussion of this distinction see Smithies [20].

of the arts" and aggregate factor supply determine in any period the ceiling level of output for that period, while the actual level attained depends on the volume of aggregate demand. As long as the level of demand output is less than the ceiling level the system works along the lines of the familiar expenditure models, but should demand output exceed the maximum output the ceiling restriction comes into play. Excess demand exists, which forces a change in the structure of the system. In the models developed below this situation is met by switching equations at the crucial point, and the system thus varies depending upon whether the ceiling has been reached or not. The switching equation device pieces together linear segments to approximate a roughly hyperbolic function.

The rate of growth of capacity output depends on the rates of growth of capital and labor and technological change. This means that given the rate of technical progress the growth of the ceiling depends on the rate of investment and on the rate of growth of the labor force; the latter in turn bears a close relation to the increase in population. The demand level of output depends on the volume of aggregate spending broken down into the main components of consumption, investment, expenditures by government, and net foreign investment.

In the present models we have introduced population, government spending, and a proxy variable for technology as the key exogenous variables. Net foreign investment has also been assumed to be exogenous, but this is of relatively minor significance. This step will doubtless meet with a certain amount of opposition, with which we are in most part sympathetic. We are aware, for example, that the growth of population in the United States may be both a result as well as a cause in the overall growth process, and that technological progress hardly falls "like the gentle rain from heaven". We did not feel, however, that we could give these problems adequate treatment. It is assumed below that changes in technology affect only the production relationship, i.e., the supply side of the system. Government outlays affect the supply side of the system to the extent that the government spends on capital goods and affect the demand side generally by influencing the volume of spending. Population is assumed to affect both demand and supply, the former through its effect on the volume of spending and the latter through its effect on the supply of labor.

Within the broad framework outlined, the process of model building takes shape immediately. On the one hand, it is necessary to deter-

mine the aggregate volume of spending, breaking that spending down into such components as may be thought desirable; on the other, to set up equations for factor demand to determine the level of employment of the factors of production.

3. THE MODEL

As indicated, output is taken to be gross national product, while the factors of production are taken to be labor and capital, the latter including all forms of capital whether privately or publicly owned except for foreign investments, land, and monetary gold and silver. Labor is measured in man-hours worked including the work of the armed forces.

A simple linear combination of these two factors is hardly adequate for interpreting the growth of output over the sample period. One would suppose that during the last fifty years or so substantial changes in technology have had their effect in increasing the overall productivity of factor inputs, for both output per man-hour and output per unit of capital employed have risen markedly[5]. It goes without saying that the problem of measuring or accounting for this technological change is fraught with difficulty, and a thorough job falls outside the scope of this paper[6]. It has been assumed that technical change has been essentially neutral in character and that it has taken the form of upward shifts in the constant term of the linear production function. The problem is to determine the extent of that shift.

The simplest way of approximating the shift is to fit a trend together with the factor inputs. The objection to this procedure, apart from the fact that it represents a confession of ignorance, is that it fails to take account of the possible ebb and flow of technical change, and assumes that it spreads over the years at a constant rate. Therefore, one is prompted to look for other variables that might be used. Two possibilities were considered, the use of some measure involving patents and a measure of education. The first case, a proxy for the state of technical knowledge, was obtained by cumulating patents granted. Once a patent is taken out the knowledge it involves passes permanently into the "state of the arts". The second measure, which was the number of school days attended, is a measure of the diffusion of technical knowledge.

[5] The decline in the overall capital-output ratio is the more interesting in view of the assumption sometimes made about the constant proportions of capital to output, and the great efforts that have gone into explaining its constancy.

[6] Professor Solow's paper [21] represents one of the few contributions in the field.

Many of the difficulties involved in using these measures are obvious ones. The usefulness of the patents measure depends very much on the extent to which patent protection is sought, and there seems to be some evidence that such protection is coming to be less in demand. Futhermore, the measure weights each patent equally, a very dubious procedure. As for the measure of education, it makes no allowance for the distinction between deepening and widening aspects in a growing population.

Equations (3.1) through (3.3) are the precise forms which were estimated.

$$(3.1) \qquad (Y + D)_t = \alpha_0 + \alpha_1 K_t^* + \alpha_2 (N_w h_w)_t + \alpha_3 t + u_{1t} ,$$

$$(3.2) \qquad (Y + D)_t = \alpha'_0 + \alpha'_1 K_t^* + \alpha'_2 (N_w h_w)_t + \alpha'_3 E_t + u_{2t} ,$$

$$(3.3) \qquad (Y + D)_t = \alpha''_0 + \alpha''_1 K_t^* + \alpha''_2 (N_w h_w)_t + \alpha''_3 T_t + u_{3t} .$$

where Y = net national product in 1929 dollars,
D = depreciation in 1929 dollars,
K^* = employed stock of capital in 1929 dollars,
$(N_w h_w)$ = man-hours worked,
t = time,
E = school days attended,
T = cumulated stock of patents,
u = random disturbance.

Depreciation or capital consumption is assumed to depend on the level of net national product and the current stock of equipment. Thus we have

$$(3.4) \qquad D_t = \beta_0 + \beta_1 Y_t + \beta_2 K_t + u_{4t} ,$$

where

K = total stock of capital in 1929 dollars.

In the absence of factor prices (excluded by our terms of reference) there appears to be no satisfactory way of determining the demand for capital services. It will be clear later that as long as the demand level of output is below the capacity or ceiling level, the production function sets the labor requirement, with the demand for capital determined independently. An empirical solution was adopted by relating the level of employed capital to the existing total stock (the supply) and the current level of gross national product (the demand) which gives the relation

$$K_t^* = \gamma_0 + \gamma_1(Y + D)_t + \gamma_2 K_t + u_{4t}, \tag{3.5}$$

where $K_t^* \leq K_t$.

To complete the supply side of our models we require an equation for the supply of labor. The main force affecting the long run growth of the labor input has undoubtedly been the increase in the level of population. In the short run, variations in the real wage level may have some effect on changes in the labor force, although the expected direction of influence is by no means clear.

Finally, we introduce a trend to adjust the influence of the population variable for changes in average man-hours worked per week.

$$(N_f h_f)_t = \delta_0 + \delta_1 N_t + \delta_2 w_t + \delta_3 t + u_{5t}, \tag{3.6}$$

where

$(N_f h_f)$ = labor force in man-hours,
N = population,
w = real wage.

The expenditure side of the models presented in this paper consists, autonomous factors apart, of the consumption and investment equations. There are two identities

$$(Y + D)_t = C_t + I_t + G_t + F_t, \tag{3.7}$$

where

C = consumption in 1929 dollars,
I = gross private domestic investment in 1929 dollars,
G = government expenditures in 1929 dollars,
F = net foreign investment in 1929 dollars,

$$K_t = K_{t-1} + I_t + (I_g)_t - D_t, \tag{3.8}$$

where

I_g = government investment expenditures in 1929 dollars.

The expenditure equations form the heart of the system examined below, as they do in nearly all of the theoretical models of cycle and growth that have been developed since the *General Theory*.

Four consumption functions were fitted to the data. Two reflected the influence of the respective Duesenberry [4] and Friedman [7] hypotheses that are more commonly referred to as the relative and permanent income hypotheses. For the relative income hypothesis we

write

$$C_t = \lambda_0 + \lambda_1 Y_t + \lambda_2 \bar{Y}_t + \lambda_3 N_t + u_{7t} , \tag{3.9}$$

where

$\bar{Y}$ = highest previous level of net national product.

To represent the permanent income hypothesis, we adopt a relation suggested by Koyck's type of distributed lag[7]

$$C_t = \lambda'_0 + \lambda'_1 Y_t + \lambda'_2 N_t + \lambda'_3 C_{t-1} + u_{8t} . \tag{3.10}$$

Liquid assets were excluded by the terms of reference, but lagged capital stock was included in one trial as a proxy for wealth[8].

$$C_t = \lambda''_0 + \lambda''_1 Y_t + \lambda''_2 N_t + \lambda''_3 K_{t-1} + u_{9t} . \tag{3.11}$$

Finally, it is interesting to compare the very simple formulation

$$C_t = \lambda'''_0 + \lambda'''_1 Y_t + \lambda'''_2 N_t + u_{10,t} . \tag{3.12}$$

The investment functions were derived from certain generalizations of the acceleration principle, embodying the belief expressed by Mr. Knox that "··· there is a strong element of truth in the acceleration principle, namely that the amount of entrepreneurs' investment is governed in some measure by a comparison of the output he can most efficiently produce with his existing plant and the output at which demand conditions now justify him in aiming."[9] A variety of functions were experimented with, but the following three proved most interesting.

$$I_t = \mu_0 + \mu_1 (Y + D)_t + \mu_2 K_{t-1} + u_{11,t}, \tag{3.13}$$

$$I_t = \mu'_0 + \mu'_1 (Y + D)_t + \mu'_2 (Y + D)_{t-1} + \mu'_3 K_{t-1} + u_{12,t} , \tag{3.14}$$

$$I_t = \mu''_0 + \mu''_1 (Y + D)_t + \mu''_2 I_{t-1} + u_{13,t} . \tag{3.15}$$

The first of these is the standard form of the generalized accelerator,

[7] For the relation between this function and the permanent income hypothesis see the paper by Klein [15]

[8] There is a great deal more to be said about asset effects that cannot be taken up here. Professor Duesenberry would like to include the stock of capital as a proxy to take into account distributive effects. Compare [5].

[9] Knox [11] p. 292. Economy of space prevents an extended discussion of the acceleration principle and the derivation of the functions employed in this paper. Knox [11] has admirably surveyed the field, and his paper provides the key references before 1952. Further general discussion may be found in the recent works of Matthews [17] and Duesenberry [5].

where it is assumed that $0 < \mu_4 < 1$. The second is a modified version of this relation with a different transformation from expected to *ex post* data, while the third reflects an adaptation of a "permanent profit" hypothesis derived from an initial distributed lag function of the Koyck type.

A more detailed study of the theory underlying (3.13)–(3.15) would reveal that it is quite likely that expectational and monetary phenomena have been inadequately treated. To the extent that expectations are formed independently of the explanatory variables, the sample residuals are likely to exhibit a cyclical pattern. In good times expectations will tend to increase the volume of investment over and above the level justified by the explanatory variables, and *vice-versa* in depression periods. The cyclical role played by monetary phenomena is excluded by the terms of reference, so that putting these two things together, there is an a priori expectation that the residuals of the investment equations will reflect certain systematic elements as well as purely random disturbances.

4. NUMERICAL ESTIMATES

In this section we present the results of statistical estimation of the relationships presented above. Initially, screening of the equations was carried out using the familar technique of classical least squares. However, once the equations presented in this paper were decided upon, the parameters were reestimated using the method of two-stage least-squares that has been developed and popularized by Professor Theil [22]. Briefly, this method involves estimating the least-squares regression of each endogenous variable in the system on a set of predetermined variables.[19] Computed values of the endogenous variables from these unrestricted reduced forms are then substituted back into the original structural equations, and the parameters of these equations are then estimated by re-applying the least-squares technique. Estimates obtained in this manner have the property of consistency that is lacking in the case of least-squares applied to individual equations within an interdependent economic system.

The estimates of parameters were based on a sample of observations, 1909–1951 by years. The war years 1917–1919 and 1942–1945 were excluded on the grounds that these were exceptional periods whose

[19] The predetermined variables used in this study were the level of tax and nontax payments, lagged gross national product, population, government spending, school days attended, lagged consumption, lagged excess capacity ($K_{t-1} - K^*_{t-1}$), cumulated patents and lagged nonwage income.

inclusion might yield untenable results in peace-time years. Sampling errors and coefficients of multiple determination corrected for degrees of freedom are presented for the equations given below. The value of the statistic δ^2/S^2 (Neumann ratio) is also given for each equation; the breaks in the sample period (in time) necessitated a minor adjustment to the method of calculating this measure.

Estimates of the production functions (3.1) and (3.3) led to reasonable results, but estimates of (3.2) did not. The introduction of the education measure turned the coefficient on labor input negative. The precise results obtained with (3.1) and (3.3) were

$$(4.1)\qquad (Y+D)_t = \underset{(4.15)}{-78.69} + \underset{(0.017)}{0.311}K_t^* + \underset{(0.076)}{0.560}(N_e h_e)_t + \underset{(0.006)}{0.281}T\,,$$

$$\bar{R}^2 = 0.95,\ \delta^2/S^2 = 0.83\,.$$

$$(4.2)\qquad (Y+D)_t = \underset{(4.16)}{-80.90} + \underset{(0.018)}{0.262}K_t^* + \underset{(0.078)}{0.697}(N_e h_e)_t + \underset{(0.029)}{1.33}t\,,$$

$$\bar{R}^2 = 0.95,\ \delta^2/S^2 = 0.81\,.$$

As might have been suspected, goodness-of-fit is relatively high, but the low sampling errors must be viewed with a certain amount of caution as guides to the stability of the estimates. Multicollinearity is present between capital and labor, which is masked by the goodness of the fit, and so is not reflected in the sampling errors. Further, the value of the Neumann ratio indicates a high probability of first order autocorrelation in the residuals. It is possible that this autocorrelation is due to the fitting of a linear form to what may in fact be a nonlinear relation. In view of this suspicion and the relatively good fit no attempt was made to improve the estimates by applying some appropriate tranformation and reestimation procedure.

Estimation of the depreciation equation gave the result

$$(4.3)\qquad D_t = \underset{(1.69)}{-6.28} + \underset{(0.016)}{0.027}Y_t + \underset{(0.010)}{0.045}K_t\,,\qquad \bar{R}^2 = 0.96,\ \delta^2/S^2 = 0.65\,.$$

The capital utilization equation was estimated to be

$$(4.4)\qquad K_t^* = \underset{(1.82)}{75.98} + \underset{(0.013)}{0.36}K_t + \underset{(0.017)}{0.98}\,(Y+D)_t\,,$$

$$\bar{R}^2 = 0.89,\ \delta^2/S^2 = 0.43\,.$$

Again, the fits are quite good, but the degree of autocorrelation in residuals is marked. As in the case of the production function, we suspect that this arises from the use of a linear approximation rather than from the omission of key variables.

In the case of the labor supply equation, the real wage added

nothing to the equation, and the trend took on the wrong sign, leaving us with the simple relation

$$(4.5) \qquad (N_f h_f)_t = \underset{(7.77)}{45.75} + \underset{(0.126)}{0.638} N_t , \qquad \bar{R}^2 = 0.90, \ \delta^2/S^2 = 0.99 .$$

Estimating the consumption functions we found the following equations:

$$(4.6) \qquad C_t = \underset{(6.72)}{-22.35} + \underset{(0.049)}{0.560}\, Y_t + \underset{(0.064)}{0.054}\, \bar{Y}_t + \underset{(0.080)}{0.360} N_t ,$$

$$\bar{R}^2 = 0.99, \ \delta^2/S^2 = 2.05 ,$$

$$(4.7) \qquad C_t = \underset{(7.65)}{-18.72} + \underset{(0.108)}{0.467}\, Y_t + \underset{(0.111)}{0.301} N_t + \underset{(0.190)}{0.224} C_{t-1} ,$$

$$\bar{R}^2 = 0.99, \ \delta^2/S^2 = 2.08 ,$$

$$(4.8) \qquad C_t = \underset{(6.08)}{-24.34} + \underset{(0.044)}{0.558}\, Y_t + \underset{(0.014)}{0.276} N_t + \underset{(0.044)}{0.054} K_{t-1} ,$$

$$\bar{R}^2 = 0.98, \ \delta^2/S^2 = 2.19 ,$$

$$(4.9) \qquad C_t = \underset{(2.04)}{-30.30} + \underset{(0.011)}{0.569}\, Y_t + \underset{(0.025)}{0.454} N_t , \qquad \bar{R}^2 = 0.99, \ \delta^2/S^2 = 2.0 .$$

There are several features of these estimates that are worth commenting upon. In all cases, as we might have anticipated with time series, the fits are remarkably good, as measured by the values of $\bar{R}^2$, and in contrast to most of the supply side equations, first-order serial correlation can be ruled out with a high probability, given the sample size (36). In the case of three of the four equations the values of the propensity to consume out of current income are quite close as point estimates. If we take the sampling variances of parameters into account, we cannot say that there is any significant difference between them. The same applies if we extend this to the marginal propensity to consume of (4.7) although this is quite a different point estimate.

Another outstanding feature of the estimates is the small contribution made by the particular variables suggested by the hypotheses referred to in Section 3. The extent to which this is true can be seen immediately by simply comparing (4.9) with the other equations. Equation (4.7) offers a different set of point estimates, but they are apparently slightly less efficient, due probably to the higher degree of multicollinearity that results from including lagged consumption.

Speaking quite empirically this result seems to be due to the inclusion of population as an independent variable. The permanent and longer run effects that in many models have been absorbed by $\bar{Y}$ and C_{t-1} were soaked up by the population variable, which provides the trend in consumption around which fluctuations are explained by

variations in the level of current income.

Some idea of the predictive power of these equations may be obtained by an examination of Table 1. In Table 1, computed values of these equations are presented for the period 1952–57, which lies outside the sample period. That is to say, the true values of the variables for these years have been inserted in the right hand side. In this way we obtain a set of residuals for the post-sample period. This suggests that we might test to determine whether these equations in some sense apply to the post-sample period. A post-sample variance of residuals can be computed and this compared against the residual variance of the sample period by means of an F-test. We feel however that this procedure is not particularly useful in the present case. Tests of this kind for the consumption equations lead to the conclusion that there has been no structural change. An examination of the post-sample residuals for all the equations reveals however that they are highly serially correlated. In the early post sample years the residuals are all negative (i.e., the computed level is too high), and in the post-1955 years the reverse is true. Any test of structural change should properly take into account not merely the residual variances of different periods but also the presence of persistent patterns in the residuals. In the case of the consumption equations estimated here, we conclude that while the relative errors in the prediction of consumption in the post-sample period are gratifyingly small, the pattern of consumption is influenced by some systematic element (which may be simply structural change) that is not fully taken into account by our equations.

Estimating the investment functions developed in the last section,

TABLE 1

PREDICTED AND ACTUAL VALUES OF CONSUMPTION
ALTERNATIVE FORMULATION OF CONSUMPTION FUNCTION
(Billions of 1929 Dollars)

Year	Actual Value	Estimating Equation			
		(4.6)	(4.7)	(4.8)	(4.9)
1952	141.4	147.08	145.35	148.82	145.75
1953	147.5	152.22	150.31	152.25	150.87
1954	150.3	151.83	151.06	152.04	150.36
1955	160.9	161.24	159.59	161.71	160.19
1956	166.3	165.01	164.53	165.41	163.74
1957	170.4	168.22	168.27	168.94	167.08

we arrive at the following results:

$$(4.10)\qquad I_t = \underset{(1.68)}{21.20} + \underset{(0.014)}{0.30}(Y + D)_t - \underset{(0.038)}{0.124}K_{t-1}\,,$$

$$\bar{R}^2 = 0.59,\ \delta^2/S^2 = 1.08\,,$$

$$(4.11)\qquad I_t = \underset{(1.74)}{22.37} + \underset{(0.033)}{0.168}(Y + D)_t + \underset{(0.031)}{0.140}(Y + D)_{t-1} - \underset{(0.009)}{0.130}K_{t-1}\,,$$

$$\bar{R}^2 = 0.58,\ \delta^2/S^2 = 1.13\,,$$

$$(4.12)\qquad I_t = \underset{(1.92)}{-0.89} + \underset{(0.005)}{0.070}(Y + D)_t + \underset{(0.021)}{0.541}_{t-1}\,,$$

$$\bar{R}^2 = 0.63,\ \delta^2/S^2 = 2.11\,.$$

The first and most obvious feature of these estimates is the relatively low degree of correlation. The second is that in the case of equations (4.10) and (4.11) the probability that the residuals are autocorrelated at the first order level is large, and the value of the Neumann ratio for (4.12) suggests that the residuals in the original Koyck formulation would also show a high degree of first-order autocorrelation[11]. Both these features require comment.

The relatively poor fits may be rationalized in two ways. One interpretation of the relatively low correlations for the investment function is that the volume of investment is subject to many small influences that show no systematic behavior; investment behavior is thus erratic and subject to much random influence. The objection to this interpretation here is that, given the values of the Neumann ratio, we cannot with confidence assume that the disturbance terms are random. This is related to the second possible rationalization of the low correlations, namely that they are low because we have failed to separate out all the systematic elements.

We have already referred briefly to reasons for believing that two sets of forces may have been omitted from our formulations of the investment function, or at least they have not adequately been taken into account, namely, expectations and the monetary elements.

The residuals of equations (4.10) and (4.11) are significantly autocorrelated, and in a broad way they are consistent with the hypotheses referred to. The residuals follow a cyclical pattern, being largely positive in prosperity and negative in depression. It seems plausible therefore to conclude tentatively on the basis of our results that the generalized acceleration principle as developed in this paper is capable of explaining some but not all of the systematic behavior of invest-

[11] On the problem of autocorrelation in the estimation of a distributed lag function see Klein [13].

ment over the sample period.

We are left then with the problem of patching up our investment relationships within the restrictions of our frame of reference, in particular, to attempt to obtain random residuals for our investment function. From a technical point of view, our parameter estimates are inefficient. The rationale of our estimating procedure requires that the disturbances should be nonautocorrelated if the sampling errors are to be taken seriously.

We postulate that the current disturbance (in say (4.10)) is proportional to the disturbance of the previous period plus a disturbance that is non-autocorrelated. Using this relation we can then transform our investment function into a new equation whose disturbance term has the desired property. In particular, let

$$I_t = \alpha_0 + \alpha_1(Y + D)_t - \alpha_2 K_{t-1} + u_t ,$$

and

$$u_t = \rho u_{t-1} + w_t, \; E(w_t) = 0, \; E(w_t w_{t-i}) = 0, \; i \neq 0 .$$

The investment equation can then be rewritten

$$I_t - \rho I_{t-1} = \alpha_0(1 - \rho) + \alpha_1(Y + D)_t - \alpha_1\rho(Y + D)_{t-1} - \alpha_2 K_{t-1} + \alpha_2\rho K_{t-2} + w_t .$$

If we knew the true value of ρ we could then compute transformed variables $I'_t = (I_t - \rho I_{t-1})$, etc. and estimate the α_i. However we do not have this information so that an estimate of ρ must be obtained. In the present case an iterative procedure was followed to obtain joint estimates of ρ and the α_i[12]. An initial value of ρ was calculated from the sample residuals, substituted into the transformed equation, and the α_i were estimated. Using the new set $\hat{\alpha}_i$, we computed a new set of u's, and reestimated ρ, proceeding as before. This process was carried out until the value of ρ and the values of the $\hat{\alpha}_i$ stabilized. In the present case three iterations were pursued until the desired degree of stability was achieved[13]. The resulting final set of two-stage-least-squares estimates is given by

$$(4.13) \qquad I_t = 0.498 I_{t-1} + 7.217 + 0.265[(Y + D)_t - 0.498(Y + D)_{t-1}] - 0.088\,[(K_{t-1} - 0.498 K_{t-2}], \qquad \delta^2/S^2 = 2.37 .$$

[12] For a discussion of alternative methods of estimating ρ see Klein [12].

[13] Actually the estimates differed only marginally after the first round. The most striking feature was the stability of ρ. From the sample this was estimated to be 0.5. After three iterations its value was computed to be 0.498.

It may be observed that α_1 has declined from 0.3 to 0.265, α_2 from 0.124 to 0.088 and that the transformation has effectively reduced the serial correlation[14].

5. EXTRAPOLATION TO 1952–1957

Given the estimates presented above, we are in a position to set up complete models of the U.S. economy, complete in the sense that the number of equations and identities is equal to the number of endogenous variables. For the purpose of making a set of *ex post* predictions, and a brief examination of the dynamic properties of such systems the following two models were adopted.

Model I.

$$C_t = -18.72 + 0.467Y_t + 0.301N_t + 0.224C_{t-1} ,$$
$$I_t = 0.498I_{t-1} + 0.265[(Y + D)_t - 0.498(Y + D)_{t-1}] - 0.088[K_{t-1} - 0.498K_{t-1}] + 7.217 ,$$
$$D_t = -6.28 + 0.0273Y_t + 0.045K_t ,$$
$$(Y + D)_t = -75.90 + 0.262K_t^* + 0.697(N_w h_w)_t + 1.33t ,$$
$$K_t^* = 75.98 + 0.36K_t + 0.98(Y + D)_t ,$$
$$(N_f h_f)_t = 45.75 + 0.638N_t .$$

Model II.

The same as Model I except that the investment function is replaced by

$$I_t = -4.0 + 0.07(Y + D)_t + 0.54I_{t-1} .$$

Both models include the identities

$$K_t \equiv K_{t-1} + I_t + (I_g)_t - D_t ,$$
$$(Y + D)_t \equiv C_t + I_t + G_t + F_t .$$

As far as the choice of models was concerned, the only real questions that arose related to the choice of consumption and production function. The production function (4.2) was selected mainly because it was apparent that over time the measurement of technical knowledge by accumulated stock of patents was likely to give a substantial downward bias to the system since the practice of taking out patents apparently declined in a period of technical progress. As for the con-

[14] The form in which (4.12) has been estimated has removed the autocorrelation of the first order from disturbances. (4.11) can be treated as we did (4.10), but it was felt that nothing was to be gained in using (4.11) rather than (4.10) so that the following sections consider only (4.10) and (4.12) as possible investment functions.

sumption equations, our choice of equation (4.7) was largely dictated by preliminary computations which revealed the simple models to be very sensitive to the marginal propensity to consume out of net national product. The value of the marginal propensity to consume is of course of great importance in determining the size of the short run multiplier, and it appeared that (4.7) introduced welcome dampening into the systems considered.

From the models I and II we obtain *ex post* forecasts for the period 1952–57. For the purpose of these particular computations, depreciation was treated as exogenous in order to simplify the calculations. This does not materially affect the results, and a comparison with depreciation taken as endogenous is given for one of the models below.

Before extrapolating beyond the sample period, some account was taken of the serial correlation of disturbances. As mentioned above in the case of one of the investment equations joint estimates were made of the first order serial correlation coefficient and the parameters of the equation. In other cases where the degree of serial correlation was high, the overall correlation was also high so that the problem of efficiency was not so serious. In those cases where the value of the Neumann ratio was not considered acceptable, first order serial correlation coefficients were calculated for the relevant equations from the sample residuals, and then used to transform the equations for the purpose of forecasting. The computed serial correlation coefficients are given in Table 2.

TABLE 2

FIRST ORDER SERIAL REGRESSION COEFFICIENTS IN SELECTED EQUATIONS

Equation	First order serial regression coefficient
(4.2)	0.579
(4.3)	0.421
(4.5)	0.522
(4.11)	0.498

It is worth stressing the fact that the values in Table 2 lend no support to the practice often followed (particularly by Professor Stone) of forming relationships among first differences of variables to remove serial correlation. This procedure assumes essentially that first order serial correlation coefficients are likely to be closer to one, in general, than to any other value. In fact, from this very small sample one might guess that one-half is likely to be much nearer the mark.

It was pointed out earlier that the models discussed in this paper

have two forms, depending on the degree of capacity utilization. It can be seen that the expenditure equations of the models considered can be solved as an independent set. They provide the level of demand output, which is then put into the production function. This together with the capacity utilization equation enables the quantity of the factors employed to be determined. This procedure is, of course, valid only so long as the solution values for the employment of factors satisfy the basic restriction that the quantity of the factor employed must be less than or equal to the total amount of the factor available; i.e., the solution value of demand output must feasible if it is to be found in this way. In the event that the level of demand output is not feasible, excess demand is assumed to exist within the system, and the actual level of output is taken to be the capacity level. This necessitates suppressing one of the expenditure equations; otherwise the system is overdetermined. We chose to suppress the consumption function, in this case leaving consumption to be determined as a residual from the gross national product indentity. In practice, prices would to some extent bear the burden of allocating the scarce resources among the competing demands of consumers, private investors, and government, but it seems reasonable to assume that the burden of adjustment will fall largely on consumers through a form of forced saving rather than on either of the other two. This implies that, other things being equal, an increase in real government outlays when the economy is at the ceiling will result in a reduction in real consumption.

Ex post forecasts of net national product for the years 1952–57, presented in Table 3, are obtained from the two different models considered. Before making these forecasts adjustments were made to the constant term of the production function (4.2) above, and the distributed lag investment function (4.12). These adjustments were made on the basis of preliminary extrapolations of the individual equations over the first three post-sample years. It was clear that the level of capacity output given by the system was consistently too low. For the most part this may have been due to the more rapid rate of technological change in more recent as opposed to the pre-war years, which form the majority of the sample observations by far. The distributed lag investment function gave values that were consistently too high; so the constant term was adjusted down.

The results are set out in Table 3. The models require the use of a ceiling version to obtain feasible results for 1952 and 1953. While several post war years can be considered, in some sense, years of

TABLE 3

NET NATIONAL PRODUCT ACTUAL AND MODEL FORECAST VALUES
(Billions of 1929 Dollars)

Year	Level of net national product			Change in net national prtoduct		
	Actual	Predicted		Actual	Predicted	
		Model I	Model II		Model I	Model II
1952	185.1	186.5	185.5	–	–	–
1953	192.1	192.0	192.0	+7.0	+5.5	+6.5
1954	188.9	194.5	191.7	−3.2	+2.5	−0.3
1955	203.7	191.4	189.4	+14.8	−3.1	−2.3
1956	206.8	200.0	200.9	+3.1	+9.4	+11.5
1957	210.3	206.5	206.8	+3.5	+6.5	+5.9

excess demand, it is not so clear why the years of the early fifties should reflect excess demand more critically than, for example, the inflationary years 1955–57[15].

An outstanding feature of both models examined is their complete failure to predict the upturn in 1955. In both cases, once the system starts downward it is difficult to halt. This is perhaps hardly surprising if one considers the sample period during which estimates were made, covering the depression of the thirties following the slump of 1929. It may be observed that the more sophisticated Klein-Goldberger model also fails to predict the recovery adequately. The fact that the K-G model gets a movement at least in the right direction may perhaps be attributable to the cognizance taken in that system of the tax changes that took place in 1954. The failure of the models used here may be in part at least due to a failure to take into explicit account the effect of the so-called built in stabilizers and tax rate changes. However the performance of the K-G model leads one to suspect that more is involved[16]. The rise in gross national product in 1955 and the rapid recovery from the recession was, for the most part, quite unexpected. One possible explanation is that 1955 was a year in which there was an autonomous step forward by consumers that accelerated the volume of investment. As revealed by the figures in Table 1 all the consumption functions estimated from the sample period give results that are too low when extrapolated beyond 1955,

[15] It has been pointed out that this suggests that the price increases of the early fifties, particularly the Korean war aftermath, were largely demand-induced; whereas the 1955–57 period is a period of cost inflation. We offer no comment on this implication here.

[16] For some forecasting results in the K-G model see Goldberger [8].

It is clear that there is a persistent tendency for the models to predict low for the later years. If the constant terms of the investment functions had not been reduced on the basis of the first three post-sample years agreement would be rather better. On the whole, with the exception of 1955, the predictions are very much of the right order of magnitude in the levels.

It should be emphasized that only point forecasts are given in Table 3. Clearly these predictions are subject to error arising from the individual error variances in equations and the sampling errors associated with the estimated parameters. Standard errors of forecast could be computed and applied to the point forecasts of Table 3. Such standard errors have not been computed in this study.

6. DYNAMICS OF THE MODELS

Exact (i.e. nonstochastic) models embodying the theoretical relations underlying this study are dynamic by virtue of the existence of lagged endogenous variables. In the discrete treatment of such models, complete systems can be reduced to a high order difference equation whose solution can be obtained analytically, and thus the dynamic character of these systems can be studied.

In stochastic models the actual path of output over time, by definition, does not reflect the influence of the exact or systematic components alone. It has been customary however to examine the behavior of econometric systems at the point where the disturbances are assumed to take on their mean (zero) value[17]. While Goldberger demonstrates that the systematic component of the Klein-Goldberger model is stable, the Adelmans [1] have shown empirically that the model is capable of generating plausible cycles if the underlying stochastic mechanism is taken into account, by drawing samples of disturbances and carrying them along over time.

The basic result here, that cycles generally result from an experiment of this kind, is hardly in doubt and is long established. The interesting feature, it would appear, is that the parameters of the Klein-Goldberger model are such that the disturbances are cumulated to generate a cycle of roughly the right sort of period and amplitude.

It is worth noting that the autocorrelation structure of an individual equation should, in general, be taken into account when examining its behavior, so that the remaining disturbance term is nonautocorrelated. In the case of second-order stochastic difference equations (for example)

[17] Professor Goldberger's recent study of the Klein-Goldberger model [8] is a case in point.

where first-order serial correlation exists in the disturbance, the equation can be transformed into one of the third order, and the time path of the variable will alter accordingly. But this transformation will not affect the qualitative behavior of the variable (e.g., whether it exhibits damped behavior or not) provided the first-order serial regression coefficient lies between 0 and 1, i.e., the disturbance itself follows a damped scheme. Thus it is apparent that a stable first-order autocorrelation structure, which is all that has been considered in this paper, will not affect the ultimate behavior of the models. Nonetheless since the models were transformed for the purpose of prediction, some dynamic behavior has been investigated in this form.

Despite the fact that the omission of the disturbance terms may in one view lead to inadequate understanding of the dynamic models of the kind considered here, the behavior of the exact part of the system remains of interest. It is still important to see whether the systematic components are capable of producing a particular dynamic behavior pattern, for many propositions are formulated in exact terms,

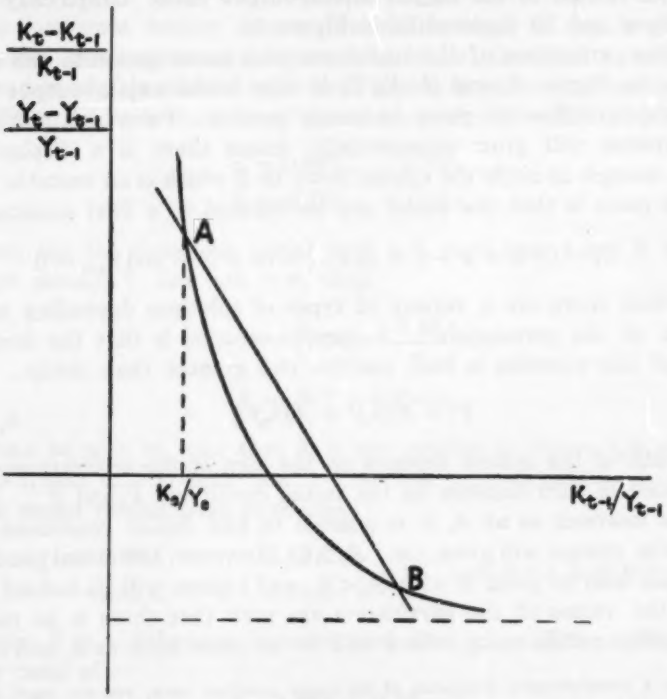

FIGURE 1

upon which light may be thrown.

It is interesting to compare the models of this paper with the simple model of Duesenberry [5].

$$C_t = aY_{t-1},$$
$$K_t - K_{t-1} = \beta Y_{t-1} - \alpha K_{t-1},$$
$$Y_t = C_t + K_t - K_{t-1}.$$

Here all the variables are *net*, and measured as deviations from their equilibrium values. From these equations we can derive

$$\frac{Y_t - Y_{t-1}}{Y_{t-1}} = (a + \beta - 1) - \alpha \frac{K_{t-1}}{Y_{t-1}},$$
$$\frac{K_t - K_{t-1}}{K_{t-1}} = -\alpha + \beta \frac{Y_{t-1}}{K_{t-1}}.$$

Thus the rate of growth in income, and the rate of growth of capital are both related to the lagged capital-output ratio. Graphically these equations may be depicted as in Figure 1.

If the parameters of the model are such as to generate the curves drawn in Figure 1, and if K_0/Y_0 is the initial capital-output ratio, the system will settle down to steady growth. Point A is stable, and the system will grow exponentially unless there is a displacement large enough to shift the system down to B which is an unstable point.

The point is that the model can be reduced to a final equation

$$Y_t - (-\alpha + a + 1 + \beta)Y_{t-1} + (a + \beta - a\alpha)Y_{t-2} = 0$$

for which there are a variety of types of solutions depending on the values of the parameters[19]. A possible solution is that the dominant root of this equation is real, positive and greater than unity. Then

$$Y_t = A_1(\lambda_1)^t + A_2(\lambda_2)^t, \qquad \lambda_1 > 1.$$

The path of the system depends on the sign of the arbitrary constant A_1 which in turn depends on the initial conditions Y_0 and Y_{-1}. If the curves intersect as at A, it is possible to find initial conditions such that the system will grow, i.e. $A_1 > 0$. However, the initial conditions can also lead to point B with $A_1 < 0$, and income will go toward zero.

If the values of the parameters are such that there is no positive root greater than unity, there will be no point such as A, and steady

[19] For a comprehensive discussion of the issues involved here, see the paper by Mr Pasinetti [18].

growth upward will not be possible. Other solutions of the final autoregressive equation are obviously possible, however Duesenberry has suggested that 'reasonable' values of the parameters for the United States are such that growth is likely to ensue, so that the multiplier-accelerator mechanism becomes the engine of economic growth rather than of cyclical fluctuations.

If we disregard the serial correlation problem for the moment and assume that depreciation is exogenous, the simple expenditure models of this paper can be brought within a similar frame of reference. Consider model I. We assume that the exogenous variables are constant (i.e. population and autonomous spending are stationary) and form first-differences of the system, which gives,

$$\begin{aligned} c_t &= \alpha_1 y_t + \alpha_2 c_{t-1} , \\ i_t &= \beta_1 y_t - \beta_2 k_{t-1} , \\ y_t &= c_t + i_t , \\ k_t &= k_{t-1} + i_t . \end{aligned}$$

The lowercase letters denote first differences. If steady growth is possible, given constant exogenous variables, the final autoregressive equation of this system must have a root greater than unity.

We write down the reduced equations, using the estimated parameters

$$\begin{aligned} c_t &= 0.66 c_{t-1} - 0.12 k_{t-1} , \\ k_t &= 0.70 k_{t-1} + 0.28 c_{t-1} . \end{aligned}$$

Now, can we choose an initial ratio c_0/k_0 such that c and k will both grow steadily? Let $c_0/k_0 = \sigma$, then

$$c_t = \left(0.66 - \frac{0.12}{\sigma}\right) c_0 ,$$

$$k_t = (0.7 + 0.28\sigma) k_0 .$$

It can be seen at once that it is not possible to choose c_0/k_0 such that $c_t > c_0$ and $k_t > k_0$ both hold. The final autoregressive equation of this model results from expanding

$$\begin{vmatrix} 1 - 0.66E^{-1} & 0.12E^{-1} \\ -0.28E^{-1} & 1 - 0.70E^{-1} \end{vmatrix} = 1 - 1.36E^{-1} + 0.49E^{-2} ,$$

where E is a difference operator such that $c_{t-1} = E^{-1} c_t$. We require the roots of

$$\lambda^2 - 1.36\lambda + 0.49 = 0 .$$

It is apparent at a glance that the model gives rise to damped oscillations and is not capable of yielding steady growth.

Suppose that the investment function is now changed to that of model II.

$$i_t = \beta_1 y_t + \beta_2 i_{t-1} .$$

The reduced equations become

$$y_t = 0.49 y_{t-1} + 0.68 i_{t-1} ,$$
$$i_t = 0.032 y_{t-1} + 0.43 i_{t-1} .$$

Again steady growth is not possible. In this case we can choose i_0/y_0 such that $i_t > i_0$ and $y_t > y_0$. But this is not sufficient. In this case we must take the roots of

$$\lambda^2 - 0.92\lambda + 0.18 = 0 .$$

This has two real roots, the dominant one being $\lambda = 0.63$ approximately.

We can change the consumption function and adopt the version

$$c_t = \alpha_1 y_t + \alpha_2 k_{t-1} .$$

The reduced equations of the new system with the investment function of model II are

$$i_t = 0.63 i_{t-1} + 0.01 k_{t-1} ,$$
$$k_t = 0.63 i_{t-1} + 1.01 k_{t-1} .$$

This system is balanced on the edge of growth, for if we take the roots of

$$\lambda^2 - 1.64\lambda + 0.63 = 0 ,$$

the dominant root is fractionally over unity[19], but is insufficiently so for the case to be clear cut.

We now drop the assumption that depreciation is exogenous, introduce the appropriate serial correlation transformations, and derive the final autoregressive equations for the forecasting versions. For model II we find the roots of

$$\lambda^4 - 2.31\lambda^3 + 2.00\lambda^2 - 0.73\lambda + 0.09 = 0 ,$$

and for model I

$$\lambda^4 - 2.28\lambda^3 + 1.95\lambda^2 - 0.73\lambda + 0.10 = 0 .$$

[19] Since the roots are functions of statistically estimated parameters they are, of course, subject to sampling error. Under certain approximations it is possible to compute these errors, but this has not been done in this paper.

The similarity of these two equations is quite striking and they yield roughly the same roots of the order $\lambda_1 = 0.3$, $\lambda_2 = 0.6$, and a pair of complex roots with modulus less than unity. Both models, therefore, are not capable of giving sustained growth[20].

So far we have discussed only expenditure models, and something remains to be said about the ceiling version of the system. In view of the similarity of the dynamic behavior of models I and II in their expenditure form, only model II was considered. Using the investment function of model II the final autoregressive equation at the ceiling was obtained. Behavior again is damped, although one of the roots is probably not significantly different from unity. This implies that the growth of the economy at the ceiling is governed by the rates of growth of population, autonomous spending, and technology. The importance of population can be briefly indicated by the following simple experiment.

Reduced form equations were obtained for model II, i.e., the endogenous variables Y, I, C, D, K, $N_w h_w$ and $N_f h_f$ were expressed as functions of their appropriate lagged values and the truly exogenous variables. There are two sets of reduced form equations, one for the ceiling and one for the expenditure version. By choosing initial values for the predetermined (lagged endogenous) variables, the system can be started up and the path of output traced. The advantage of doing this from the reduced form is that the contributions of the exogenous and predetermined variables are isolated. For example in the expenditure version the short run multiplier of population with respect to net national product is found to be 0.613 and the multiplier on government spending 2.044. These of course are the computed values of $\partial Y/\partial N$ and $\partial Y/\partial G$, holding all predetermined and the exogenous variables (I_g, F) constant[21].

[20] It is worth noting that the real roots of these equations give

$$\lambda^2 - 0.9\lambda + 0.18,$$

which is approximately the appropriate equation for Model II with depreciation exogenous. The effect of adding depreciation as an endogenous variable is thus to "add" a cycle that is highly damped. It is not possible to identify this with a replacement cycle for fairly obvious reasons.

[21] Extrapolation from the reduced form equations (including D as endogenous) led to the results

	Predicted NNP	*Actual NNP*
1952	185.7	185.1
1953	191.6	192.1
1954	193.0	188.9
1955	191.9	203.7
1956	203.3	206.8
1957	208.3	210.3

Compare with Table 3 above.

Initial values were taken from 1952 and inserted into the reduced form equation. The path of gross national product was then computed under the following assumptions. The value of total autonomous spending was kept fixed at its 1952 level. Population was allowed to follow its time path (increasing at the rate of about 3 million a year) and t of course took on the appropriate integral values. A second set of computations were made holding population constant.

The results are graphed in Figures 2a and 2b. Given the actual growth in population, gross national product grows at a decreasing rate. Excess demand exists but is not removed over the five-year period (i.e. the desired expenditure level of output exceeds the ceiling level of output over that period). This means that the actual level of gross national product is the ceiling level. If we now assume that population remains constant gross national product still grows over the five-year period, but its rate of growth is substantially reduced, and after three years excess demand is eliminated, the projected level of gross national product becoming feasible[22].

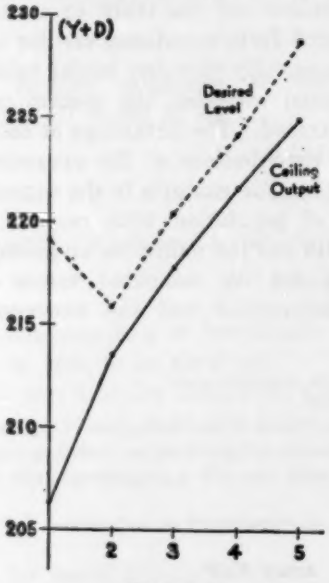

FIGURE 2a

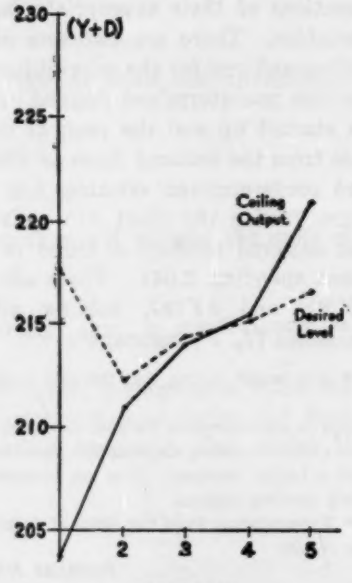

FIGURE 2b

[22] It should be emphasized here that despite the adjustment to the constant term of the production function the capacity level of output seems to run much too low, taking the period as a whole. However, even with a higher ceiling the result *vis-à-vis* the importance of population is unlikely to be substantially affected.

The model considered here at the ceiling illustrates at least four different points. First, the parametric structure is such that it is possible for the system to "crawl" along the ceiling for several periods; i.e., there is no inherent tendency for the system to bounce back immediately, provided that the initial excess demand is sufficiently large. Initial conditions here play an important role. Secondly, the computations illustrate the slow-moving character of the economy as depicted by model II. We know that the model is damped and that the rate of growth of output will decline, but this decline takes place relatively slowly. In the case where population follows its actual course the rate of growth declines from about 2.9 percent to 1.7 persent over a five year period. If population is held constant the decline is rapid in the first-period but then is relatively slow. In the third place, the model considered illustrates the fundamental interdependence of the rate of growth of the ceiling and expenditure levels of output. The ceiling level of output is not a *deus ex machina* which simply bounds an oscillating level of desired output; the rate of growth of ceiling output is dependent on endogenous factors, and there is some evidence that in the boom the rate of growth of ceiling output is likely to be below that of the upswing. The reason for this is fairly obvious, namely, the reduction in the rate of investment as the economy hits the ceiling. This contributes to the possibility of "crawling" along the ceiling, since a declining rate of growth of desired output is met with a falling rate of growth of capacity. Finally, the computations illustrate the important role of population growth within the specification of model II.

7. SOME CONCLUSIONS

It is tempting at this point to cry halt, and to leave the reader to interpret and evaluate the results as he will. But we offer some tentative conclusions to provide a focal point for such an evaluation.

At the outset three criteria were offered by which to judge the results of this paper. They were statistical within the sample period, statistical outside the sample period, and dynamic. From a statistical point of view the results seem fairly acceptable, or probably at least as good as any other specification is likely to produce. However it must be recognized that expectational and monetary elements have been excluded and this (following the analysis of section 5) shows up in the estimation of the generalized accelerator equation of model I. It must also be emphasized that it is quite possible to formulate other models that will fit the data equally well. From a statistical point

of view, the results of estimation prove nothing—they can only be judged on the basis of their consistency with the underlying theories. This we leave to the reader.

From the point of view of prediction, the results are much better than we expected. With the exception of 1955 we get fairly sensible results, but other forecasters have found difficulty with that year and we are not alone. In view of the length of the forecast period outside the sample period, the models do not do too badly when judged from the viewpoint of prediction in the short run. However, by the end of the sample period, it is clear that reestimation adding more recent data would be desirable as, for example, the consumption and production functions are tending to run too low.

It is from the third point of view, that of dynamic character, that the models are in some ways most interesting. As we have pointed out, the critical elements in many business cycle models are the parametric values that are assumed by the authors. The models discussed in this paper, like the linearized version of the Klein-Goldberger model, are inherently stable, and are not consistent with the hypotheses of explosive oscillations or of steady growth.

The engine of growth is made up of population, autonomous spending and technical change, and should the rate of growth of these factors go to zero, the economy would cease to expand[23]. We do not regard this outcome as unreasonable, given the existence of a fairly stable distribution of income.

However, as was emphasized at the outset our results cannot directly disprove alternative theories of dynamic behavior within the economy, since the coefficients cannot be identified with the coefficients of the theoretical models. The identification problem (in this nontechnical sense) is important. It is worthwhile, however, to discuss the possible relationship of the statistical parameters to the theoretical values sometimes assumed. Duesenberry, for example, thinks that "reasonable" values of the coefficients of the model briefly discussed in section 5 lead to growth; Matthews, on the other hand, feels that "reasonable" values of the coefficients of the multiplier-accelerator equation lead to explosive oscillation. The sting in the tail is how reasonable is "reasonable."

It is useful to approach this problem by considering a simple example. Assume the model

[23] We do not however regard this as a likely possibility, mainly because of a persistent belief in the possibilities of technical advance in both production and consumption. This latter aspect has been ignored in this paper, but we feel that it is very important. Product innovation is an aspect of growth frequently neglected.

$$S_t = sY_t ,$$
$$I_t = \sigma(Y_t - Y_{t-1}) ,$$
$$S_t = I_t ,$$

where S, I, and Y are net saving, net investment and net income, respectively. The final autoregressive equation of this system is

$$Y_t + \frac{\sigma}{s - \sigma} Y_{t-1} = 0 .$$

For this system to grow, it is required simply that $\sigma > s$ and $s > 0$. Suppose that we decide to apply this model to the statistical data for the U.S. economy over the sample period, and in particular fit the investment function in the form,

$$K_t = \sigma Y_t .$$

Then it is hardly necessary to look at the data before deciding that the statistically estimated model will produce growth. In fact it will be found that σ is something between 2 and 3 and s is about 0.15 (the average propensity to save). Now, suppose we modify the model and write

$$C_t = cY_{t-1} ,$$
$$K_t = \sigma Y_{t-1} ,$$
$$Y_t = C_t + K_t - K_{t-1} ,$$

where C is consumption. The final autoregressive equation of this system is

$$Y_t - (c + \sigma)Y_{t-1} + \sigma Y_{t-2} = 0 .$$

This system will have explosive oscillations if $(c + \sigma)^2 - 4\sigma < 0$ and $\sqrt{\sigma} > 1$. If we obtain statistical estimates of σ and c from the annual data we might not be surprised to obtain values of the order $c = 0.8$, $\sigma = 2$, which satisfy this condition. This would appear to be Mr. Matthews vindicated.

The point that is worth making with regard to these simple models, is that while the proponents, discussants, and model builders dealing in the trade of these simple systems have not usually performed statistical analysis of this simple variety explicitly, they have done so implicitly, obtaining some rough idea of the average saving-income and the average capital-output ratios from time series data. They have not conjured figures out of thin air; they have paid some attention to the empirical background. "Reasonable" seems to mean something

like average values, or average ratios.[24]

Our own statistical study has been very simple, and only a handful of relationships have been involved. However, we have attempted to do a little better in obtaining parameter estimates by taking into account some of the exogenous factors. Our estimates differ from the class of "reasonable" values in implying a rather higher short run propensity to save out of income and in producing parameters of the investment equation that seem to indicate a degree of quiescence that the acceleration theorists might regard as suspicious.

There are two points that should be recalled immediately. One is that investment for the purpose of statistical analysis has been defined *gross* and not *net* as is customary in most models, and the other that gross private investment includes all types of investment, plant and equipment, residential construction, and inventories. The first point is relevant in so far as there is a presumption that gross investment is likely to be more stable than net, in relation to changes in output.[25]

The relevance of the second point is obvious. Critics of the acceleration principle have sometimes adopted the view, not that it does not apply, but that its application is strictly limited. To some extent the defenders of the principle have got round this by making the rather nebulous distinction between autonomous and induced investment. The former being independent of short-run changes in the level of output. The critics however would insist that investment induced in this way is likely to be a very small part of total investment, so that the role of the acceleration principle in explaining overall investment will be a minor one. Certainly our statistical estimates are affected by the inclusion of residential construction. To the extent that there is a strong corpus of autonomous or longer range investment included in our data, the relation between investment and current output will be damped down. Thus both the degree of aggregation in the data, and the absence of any distinction between induced and autonomous investment work in the direction of damping down the link between current gross investment as we have defined it, and changes in the output level.

It may be observed that the relation between the volume of investment and the difference between the desired and beginning period capital stocks, is governed by the inverse of the length of the gesta-

[24] For some empirical discussion of the "great ratios" of economics see Klein and Kosobud [14]. They follow a procedure something like that of the first simple model and, of course, get growth.

[25] This presumption is founded on an assumed stability in replacement investment, the defense of which would take up an inordinate amount of space at this point.

tion period as well as by frictional factors due to inertia and delays on the part of entrepreneurs. These factors together with the principle of profitability all serve to break or at least weaken the physical link between investment and output changes, which has served as the rationale of the naive accelerator.

It is clear that there are several possible reasons why the statistically estimated parameters of an investment function (in particular that of model I) are such as to give rise to damped rather than explosive behavior, and why they appear to bear little or no relation to the values that the naive accelerator theories might lead us to expect. However, the results are at least suggestive of the fact that it may be correct to think that the acceleration principle in a physical sense only has limited application (otherwise it would tend to dominate more). The generalized accelerator may have much wider application, but here the profit notion and expectational phenomena may lead to a "true" value of the accelerator coefficient that is well below the "reasonable" values that tend to be inserted into simple models. Clearly, there is much scope for further investigation, disaggregating investment into suitable components and perhaps attempting to draw a distinction between induced and autonomous investment.[35]

In this paper we have confined ourselves to the modest task of interpreting the data from 1909–1951 in terms of a set of simple hypotheses without disaggregation. The models we have considered fit the data reasonably well and give year-to-year predictions that are fairly sensible. But the models are stable and cannot give rise to fluctuations without further additional factors being introduced. We have already indicated that the monetary and expectational elements have not been properly handled within the framework of our hypotheses. These factors, particularly expectations, are likely in our view to prove significant. There is, moreover, the further point that the Adelmans' study has underscored that the models we have dealt with are stochastic, and that our study of their dynamic character has been undertaken on the assumption of zero disturbances. It is possible that if these disturbances were taken into account in a manner similar to that adopted by the Adelmans, oscillations might be generated after

[35] Fisher [6] fitted the simple Hicks model to data for the U.S., 1929–41. He distinguishes between autonomous and induced investment, and concludes that the result is consistent with Hicks' explosive case. However, there are some limitations to Fisher's analysis: a) The small size of sample, b) a rather doubtful version of the consumption function applied to annual data, c) an equally unsatisfactory form of the investment function as applied to annual data. The criticisms raise an issue which we have not treated, namely the problem of aggregation over time.

the same fashion as those generated by the Klein-Goldberger model. There is room for much further research, both theoretical and applied, in this direction.

We are inclined to suggest somewhat tentatively that our results justify study of the generalized multiplier-accelerator model applied to *ex post* data. The results presented in this paper can only be regarded as preliminary to more extensive statistical analysis involving the construction of more sophisticated models. In some ways the results suggest more questions than they answer. At most they may provide a focal point for further examination of the relationship of *ex post* data to the hypotheses formulated in this area by the theoretical economists.

University of Manchester, England, and
Haverford College, U.S.A.

APPENDIX

TABLE A-1

BASIC DATA

(Money Values in Billions of 1929 Dollars)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Year	Net national product	Depreciation	Consumption	Gross private domestic investment	Government	Government investment	Net foreign investment
	Y	D	C	I	G	I_g	F
1909	49.9	5.9	41.3	11.1	3.8	a	−.3
1910	50.3	6.2	42.0	10.8	3.9	a	−.3
1911	51.9	6.4	44.1	9.6	4.6	a	.1
1912	54.7	6.6	45.2	11.2	4.6	a	.1
1913	56.6	6.9	46.7	12.1	4.5	a	.3
1914	51.5	7.1	46.1	7.8	4.8	a	−.2
1915	53.1	7.3	45.3	7.7	5.0	a	2.5
1916	61.4	7.5	49.4	10.7	4.7	a	4.0
1920	64.7	8.6	52.7	12.7	5.6	a	2.3
1921	63.4	8.2	59.1	7.5	6.5	a	1.5
1922	67.1	8.7	58.1	10.7	6.4	a	.6
1923	76.9	8.9	63.4	15.6	6.3	a	.5
1924	79.4	9.0	68.1	12.3	6.9	a	.1
1925	81.1	9.4	66.1	16.4	7.4	a	.7
1926	86.3	10.1	71.5	17.1	7.4	a	.4
1927	87.1	10.2	73.1	15.6	7.9	a	.7
1928	87.9	10.6	74.8	14.5	8.2	a	.1
1929	93.4	11.0	78.9	16.2	8.5	a	.8
1930	84.5	10.9	74.6	10.4	9.4	a	.6
1931	78.8	11.7	72.5	6.8	10.0	a	.2
1932	66.2	10.2	66.0	.8	9.5	a	.1
1933	64.2	10.0	64.6	.4	9.4	a	−.1
1934	70.8	10.0	68.0	1.9	10.9	a	.1
1935	81.2	10.2	72.3	8.8	11.0	a	−.6
1936	90.3	10.6	79.7	9.3	12.7	a	−.7
1937	98.2	10.9	82.6	5.5	12.3	a	.4
1938	92.3	10.9	81.3	6.8	14.0	a	.1
1939	99.9	11.0	85.9	9.9	14.4	a	.8
1940	109.6	11.4	90.4	14.2	15.0	a	1.2
1941	126.3	12.4	96.4	18.8	23.0	a	.4
1946	150.9	14.7	120.9	21.5	20.1	a	3.1
1947	147.5	16.6	122.6	18.2	17.5	a	5.7
1948	155.0	18.0	125.2	24.9	21.9	a	1.1
1949	151.6	19.0	128.5	16.6	25.3	a	.2
1950	167.6	19.8	136.3	28.9	23.4	a	−1.2
1951	179.2	21.2	137.2	29.1	32.0	a	1.1
1952	185.1	b	141.4	24.4	39.2	11.8	.8
1953	192.1	b	147.5	23.5	43.0	12.6	−.1
1954	188.9	b	150.3	22.8	37.8	11.2	.6
1955	203.7	b	160.9	29.1	36.6	10.1	.6
1956	206.8	b	166.3	28.7	36.5	9.9	.1
1957	210.3	b	170.4	27.3	37.9	10.7	.0

APPENDIX

TABLE A-1

BASIC DATA

(Money Values in Billions of 1929 Dollars)

	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Year	Population	Man hours Available	Man hours Employed	Capital stock Available	Capital stock Employed	Cumulated patents	Education	Personal tax + non tax payments[a]
	N	$N_f h_f$	$N_w h_w$	K	K^*	T	F	Q
1909	90.5	99.2	94.0	221.2	209.7	37.4	53.2	698
1910	92.4	102.5	96.6	227.5	214.1	73.3	54.8	723
1911	93.9	105.0	98.5	232.3	217.9	107.4	56.4	727
1912	95.3	107.3	101.6	238.6	226.1	145.1	58.1	727
1913	97.2	107.1	102.4	245.5	234.1	180.9	59.7	787
1914	99.1	109.4	100.7	248.2	228.3	222.7	61.4	804
1915	100.6	110.6	100.0	250.5	226.2	267.6	63.0	833
1916	102.0	112.3	107.0	255.9	243.6	313.5	64.6	956
1920	106.5	111.7	107.2	276.3	265.2	474.7	71.2	2234
1921	108.5	111.3	96.9	277.9	244.8	516.1	74.1	1706
1922	110.0	112.2	103.8	282.4	260.9	556.4	77.0	2020
1923	112.0	115.5	111.8	291.7	282.4	597.2	79.8	1820
1924	114.1	115.7	109.5	297.9	281.5	642.4	82.7	1908
1925	115.8	118.1	113.4	308.1	295.8	692.2	85.6	2047
1926	117.4	119.4	117.4	318.5	312.4	739.8	88.5	2121
1927	119.0	121.9	116.9	327.5	314.0	784.2	91.4	2320
1928	120.5	123.5	118.0	335.2	320.4	830.1	94.2	2796
1929	121.8	124.1	120.3	325.8	315.3	878.7	97.1	2643
1930	123.1	123.4	112.6	329.8	300.4	927.3	100.0	2620
1931	124.0	123.3	103.8	330.0	277.5	982.4	100.5	2076
1932	124.8	120.9	92.4	323.8	246.9	1039.3	101.0	1766
1933	125.6	123.1	92.6	316.8	237.9	1090.9	101.5	1931
1934	126.4	118.1	92.6	312.1	244.8	1138.6	102.0	2113
1935	127.2	122.1	97.7	314.3	251.1	1183.5	102.5	2491
1936	128.0	128.4	106.8	318.5	264.3	1228.3	103.0	2925
1937	128.8	129.9	111.4	326.9	277.8	1271.6	103.5	3656
1938	129.8	128.0	103.8	328.0	265.7	1315.1	104.0	3542
1939	130.9	140.0	108.5	332.4	275.2	1364.2	104.5	3020
1940	131.9	132.1	113.0	340.3	290.6	1413.0	105.0	3203
1941	133.1	137.5	124.2	354.7	322.4	1461.0	105.3	3468
1946	140.0	136.5	131.4	405.0	389.2	1591.2	106.7	4980
1947	143.5	134.9	130.3	410.7	395.9	1613.6	107.0	5212
1948	146.1	135.6	131.0	422.6	408.6	1641.7	107.3	5299
1949	148.7	133.6	126.4	426.5	403.0	1681.5	107.6	5235
1950	151.2	135.6	128.9	442.1	400.9	1729.5	107.9	5499
1951	153.4	140.9	136.8	460.5	446.6	1778.2	108.2	5665
1952	155.8	143.1	139.4	476.1	463.2	a	a	a
1953	158.3	144.3	140.9	490.4	478.1	a	a	a
1954	161.2	140.9	133.8	501.8		a	a	a
1955	164.3	143.7	137.9	517.4	496.7	a	a	a
1956	168.2	145.0	139.5	531.2	511.0	a	a	a
1957	171.2	144.3	138.1	543.9	520.5	a	a	a

a) Data not utilized.

b) Data not available at time of publication.

c) Values in millions of 1929 dollars.

SOURCES:

Many of the data used in this study are from John W. Kendrick, ***Productivity Trends in the United States***, a forthcoming volume to be published for the National Bureau of Economic Research. We wish to express our thanks to Professor Kendrick for giving us permission to use and publish his preliminary estimates. We also wish to express our thanks to Professor L. R. Klein and the Econometric Research Unit of the University of Pennsylvania for data which was made available to us.

Column 1: The sum of columns (3) through (7) less column (2).

Column 2: *1909–1955.* Kendrick, *op. cit.*, Table A-1, "Gross & Net National Product: Adjusted Kuznets Concepts," column 2.
1956–1957. Extrapolator: U.S. Dept. of Commerce data on Capital Consumption Allowances, shifted to Replacement Cost Basis and including Government by the *Econometric Research Unit*, University of Pennsylvania.

Column 3: *1909–1955.* Kendrick, *ibid.*, Table A-II a, "Gross National Product: Commerce Concept Derivation from Kuznets Estimates," column 4.
1956–1957. Extrapolator: U.S. Dept. of Commerce estimates of consumption in constant dollars.

Column 4: *1909–1955.* Kendrick, *ibid.*, Table II a. The sum of Gross Private Domestic Investment, Commerce Basis plus Change in Business Inventories.
1956–1957. Extrapolator: U.S. Dept. of Commerce data on Net Private Domestic Investment in constant dollars. The series extrapolated was column (4) less column (2). Column (2) was then added back.

Column 5: *1909–1955.* Kendrick, *ibid.*, Table A-II a, column 10.
1956–1957. Extrapolator: U.S. Dept. of Commerce estimates of government expenditures.

Column 6: *1952–1957.* Kendrick, *ibid.*, Table A-II a, column 6. Extrapolated by use of U.S. Dept. of Commerce estimate of government investment (the sum of government purchases of military equipment and stockpiling plus federal construction, deflated by the implicit price deflator for the federal government plus state new construction in constant dollars).

Column 7: *1909–1955.* Kendrick, *ibid.*, Table A-II a, column 9.
1956–1957. Extrapolator: U.S. Dept. of Commerce estimate of net exports in constant dollars.

Column 8: *1909–1955.* Graham and Schwartz. ***Personal Income by States since 1929***, pp. 144–145. (Millions of persons).
1956–1957. U.S. Bureau of the Census, ***Current Population Reports***, Series P-25, No. 206, p. 3. (Millions of persons).

Column 9: *1909–1954.* Column (10) multiplied by the ratio of the labor force to the number employed. The series on proportion unemployed was constructed by forming the ratio of the number of unemployed workers reported in S. Lebergott, "Annual Estimates of Unemployment in the U.S. 1900–1954," ***The Measurement and Behavior of Unemployment***, (Princeton: Princeton Univ. Press, 1957) to the labor force as reported in the *Statistical Abstract of the United States 1954*, p. 240. (Millions of man hours).
1955–1957. Labor Force and Employment as reported by the U.S. Bureau of the Census. (Millions of man hours).

Column 10: *1909–1955.* Kendrick, *ibid.*, Table A-X, "Manhours Worked," column "Total, including Military." (Millions of man hours).
1956–1957. Extrapolated using an index of hours worked prepared by the Econometric Research Unit, University of Pennsylvania. (Millions of man

hours).

Column 11: Kendrick, *ibid.*, Table A-XVI, Real Capital Stock less Monetary Gold and Silver for 1908 (214.4 billions) plus column (4) and Public Investment less column (3). Public Investment from A-II a, column 6, extended as per explanation of column (6).

Column 12: Column 11 times ratio of the number employed to the labor force derived as explained in column 9.

Column 13: The cumulated sum from 1909 of "All Patents" less "Reissues" as reported in *Historical Abstract of the U.S.* for 1909–1945, and in the *Statistical Abstract of the U.S.* thereafter. (Thousands of patents).

Column 14: Index 1930 = 100. Total number of days attended by all pupils in public day schools, continental U.S., linear interpolation and extrapolation of decennial values. Source: *Biennial Survey of Education in the United States*, 1954, p. 27.

Column 15: *1909–1955.* Kendrick, *ibid.*, Table A-II a, column 2. (Millions of dollars).

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INCOME ELASTICITY OF PUBLIC EDUCATION*

By Werner Z. Hirsch

1. INTRODUCTION

Throughout the world the need for more and better education is of greatest importance. This holds not only for the younger and economically underdeveloped nations, but even for as wealthy a country as the United States. There, too, in recent years concern has been expressed about the relatively small portion occupied in consumer budgets by public services, particularly education.[1] While the United States consumer can choose freely whether to spend his increased income mostly on food, a new car, a trip, or books and symphony tickets, when it comes to the public services his choice is only indirect. It is his money which pays for roads, parks, public education, etc., but if he wishes to choose how much of his money will go for each of these he can attempt to influence conditions only through the political process, or, under a federated political and fiscal system, he can also move to a community in which the level of the various services happens to coincide more closely with his wishes. Thus, although it is relatively easy for most people to decide what portion of their income will be spent for food, entertainment, furniture, etc., it is much harder for them to determine what part of their budget shall be apportioned for public education. How does this portion change as people's income increases? Or, in short, what is the income elasticity of public education?

This paper will attempt to make some estimates of the income elasticity of education, specifically of public primary and secondary education in the United States. Both time series and cross section data will be used for the analysis. At the end, some policy implications of the empirical findings will be explored. It is hoped that these will prove of interest elsewhere, particularly in countries also having a federated tax system.

2. UNITS OF MEASUREMENT

Income elasticty in general relates a relative change in quantity to

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[1] A good example of this concern is President John F. Kennedy's 1961 Education Message to the United States Congress, *New York Times*, February 21, 1961, p. 22, and John K. Galbraith, *The Affluent Society* (Boston: Houghton Mifflin, 1958), especially pp. 132-37.

a corresponding relative change in income. Other factors that might induce quantity changes are assumed to be held constant, or in mathematical terms, are partialled out. In a similar way, income elasticity of public education can be defined, but before measurements can be attempted, it is necessary to find appropriate units to quantify education and income.

As in the case of most services, physical quantity measurements are difficult to devise. Instead, expenditures of a meaningful unit are often used. The unit that will be used in this study is X_1, daily total current expenditures, plus debt service, for public primary and secondary education per pupil in average daily attendance (ADA). As a unit of measurement, it has the advantage of being highly uniform throughout the United States as well as over time.[2] Per capita personal income (X_4) will be the income unit.

Thus, income elasticity of public education is defined as

$$\frac{EX_1}{EX_4} = \frac{\partial X_1}{\partial X_4}\frac{X_4}{X_1} . \tag{2.1}$$

3. A TIME SERIES REGRESSION

What factors other than income are likely to bring about changes in the quantity of education and its cost, and therefore need to be quantified and their effect held constant? In this study two such factors will be included,

X_2 = per cent of public high school enrollment relative to total public school enrollment, and

X_3 = per cent of pupils (5–19 years old) in average daily attendance living in urban areas.

The first factor bears on costs since, compared to primary schools, high school classes are relatively small; equipment is relatively expensive, and teacher salaries tend to be relatively high. Likewise, urbanization can affect costs. For example, in urban communities children live relatively short distances from school and often can do without school buses. In many rural communities school enrollment is small, resulting in relatively high costs, considering the low quality of education offered.

Additional variables were considered but were ruled out after careful examination. For example, the price level of goods and services bought

[2] The same uniformity, for instance, does not exist if annual expenditures are the unit of measurement; the length of the school year varies by quite a few days, depending on district and year.

by public schools is so strongly affected by income level that the two have a very high correlation. Quality of education should not be held constant, since income changes must be permitted to find their expression in quality changes which, in turn, can then lead to expend-

TABLE 1

EXPENDITURES, HIGH SCHOOL ENROLLMENT, URBANIZATION AND INCOME, UNITED STATES SELECTED YEARS, 1900-1958

	X_1	X_2	X_3	X_4
Year	Daily total current expenditures plus debt service for primary and secondary education per pupil in ADA (dollars)	Public high school as a percent of total public enrollment	Percent of pupils in ADA (age 5-19) from urban areas	Per capita personal income (dollars)
	1	2	3	4
1900	.126	3.3	39.6	199.2
1902	.134	3.5	39.8	218.5
1910	.190	5.1	40.4	300.6
1913	.220	6.1	42.0	325.1
1920	.359	10.2	45.9	712.3
1922	.441	12.4	47.4	552.4
1930	.570	17.1	53.3	624.1
1932	.564	19.6	53.6	401.4
1940	.568	26.0	54.6	595.5
1942	.647	26.0	55.1	915.7
1946	.854	24.1	55.5	1,268.1
1948	1.102	23.6	54.3	1,435.1
1950	1.284	22.8	52.1	1,506.2
1952	1.501	22.1	51.5	1,739.0
1954	1.628	21.8	51.8	1,784.5
1956	1.834	22.1	51.7	1,965.3
1958	2.065*	22.4	51.5	2,030.3

* Estimate

Sources: Columns 1 and 2. Work sheets of historical education series collected by Vance Grant, educational statistician, Division of Statistics and Research Services: Reference, Estimates, and Projections Section, U.S. Department of Health, Education, and Welfare, Office of Education.
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1958. From *Economic Indicators*, May 1959, pp. 2,3, and 5.

iture changes.

In Table 1, expenditure, high school enrollment, urbanization and income data are given for selected years 1900–1958.

The multiple regression equation was found to have the following values for selected years during 1900–1958:

$$(3.1) \qquad X_1 = .113 + \underset{(.0241)}{.00427}\,X_2 - \underset{(.0306)}{.00520}\,X_3 + \underset{(.0000976)}{\underline{.000924}}\,X_4 .$$

The figures in brackets are standard errors of the net regression. With 13 degrees of freedom, coefficients are statistically significant at a probability level of .05 if the ratio between coefficient and standard error is larger than 2.18, neglecting the sign. The statistically significant coefficient is underlined. The coefficient of multiple determination adjusted for degrees of freedom lost ($\bar{R}^2_{1.234}$) is .969. It is highly significant at a probability level of .05. The partial correlation coefficient, $r_{14.23}$, is + .872. Thus, about 76 per cent of the 1900–1958 variation in the cost of daily per pupil current expenditure (plus debt service) can be explained on the average in terms of per capita personal income change, holding the effect of changes in the public high school-all public school enrollment ratio and urbanization constant.

An estimate of the average income elasticity of public education during 1900–1958 can be obtained by multiplying the net regression coefficient relating expenditures to personal income ($b_{14.23}$) by the ratio $\bar{X}_4/\bar{X}_1$. For 1900–1958 income elasticity of public education is estimated to be + 1.09, just slightly above unit elasticity. Thus, it can be concluded that during 1900–1958 a one per cent increase in per capita personal income was on the average associated with a 1.09 per cent increase in daily total current expenditures for public primary and secondary education per pupil in ADA. It assumes that the effect of the other variables of the equation is held constant.

4. CROSS SECTION REGRESSIONS

Income elasticities based on time series data indicate the average percentage change in per pupil education expenditures that results when average per capita personal income changes over time by one per cent. It reflects certain dynamic changes in our society. But there is another income elasticity coefficient which provides more of a static picture. It is based on cross section data and reflects the average percentage change in per pupil education expenditures that is brought about as average per capita personal income changes from one school district (or state) to the next by one per cent within a

given time period. Static income elasticities of education, based on cross section data, appear to be even smaller than the dynamic one. Three empirical attempts have been made to estimate them. For example, Solomon Fabricant, using 1942 data of the 48 states, found the income elasticity coefficient of education to be + .78.[3] In addition to 1938–1942 per capita personal income, Fabricant's analysis included two other independent variables—urbanization and density in 1940. Income elasticity of all state and local government services was + .90, but if education were excluded, it was somewhat higher, i.e., + .96. Such services as come under the heading of general control, health and hospital, highways and police were income elastic, while in addition to education, fire, public welfare, and sanitation were income inelastic.

Harvey E. Brazer estimated the income elasticity of education for 40 large cities using 1949 median family income and 1953 per capita operating expenditure data. He computed the income elasticity to be + .73.[4] Other variables included in the analysis were

> Ratio of city population to metropolitan area's population in 1950,
> Students in ADA per 1,000 of 1950 population, and
> Inter-governmental revenue per capita for education, 1953.

The present author used 1951–1952 and 1954–1955 cross section data to estimate an elasticity measure of education for the St. Louis City-County area which comes close to being an income elasticity. More correctly, it is the property elasticity of education, since average assessed valuation of real property per pupil in ADA is used in place of income. This was done because it is, at best, very difficult to find income data for school districts. Assessed valution data are readily available. There is some evidence that in the St. Louis City-County area, assessed valuation of real property per pupil in ADA and median income are highly correlated. For example, the rank correlation coefficient, relating median income (1949) and 1954–1955 per pupil assessed valuation in ten districts coterminous with municipal or census tract boundaries, is + .905. It is statistically significant at a probability level of .05.

In addition to X_4, average assessed valuation of real property per

[3] Solomon Fabricant, *The Trend of Government Activity in the United States Since 1900* (New York: National Bureau of Economic Research, Inc., 1952), 125.

[4] Harvey E. Brazer, *City Expenditures in the United States*, Occasional Paper 66, (New York: National Bureau of Economic Research, Inc., 1959), 58.

TABLE 2

EXPENDITURES, NUMBER OF PUPILS, HIGH SCHOOL PUPILS, DENSITY, GROWTH AND PROPERTY VALUE, ST. LOUIS CITY-COUNTY AREA 1951–1952 AND 1954–1955

		X_1	X_4	X_5	X_6	X_7	X_8
		Annual total current expenditures plus debt service for public primary and secondary education per pupil in ADA	Average assessed valuation of real property per pupil in ADA	Number of pupils in ADA in public primary and secondary schools	Public high school pupils as a per cent of total pupils in ADA	Number of public school pupils in ADA per square mile	Per cent increase in public school pupils in ADA, 1951–1955
	Number	1	2	3	4	5	6
1951–1952	1	243.29	7,055	1,704	22.7	189.3	52.1
	2	275.15	8,281	657	31.8	82.1	58.0
	3	198.11	5,179	1,163	16.9	145.4	110.9
	4	277.00	7,674	1,366	17.0	341.5	29.9
	5	561.76	19,977	2,007	34.9	699.0	6.8
	6	198.30	5,507	1,573	26.7	15.7	56.6
	7	236.23	6,155	2,918	19.9	194.5	73.5
	8	257.86	5,975	1,477	26.0	492.3	14.2
	9	316.68	26,083	532	0.0	9.7	225.4
	10	284.42	9,336	1,060	24.7	212.0	28.0
	11	132.38	1,553	1,013	17.6	1,013.3	26.3
	12	239.55	6,590	4,407	21.6	244.8	40.1
	13	549.14	30,118	1,020	0.0	56.7	157.5
	14	324.04	9,354	2,494	25.7	623.5	11.1
	15	137.65	3,732	486	20.5	97.2	34.6
	16	344.64	8,070	4,867	24.7	486.7	18.8
	17	143.49	3,713	1,717	17.2	90.4	40.4
	18	221.82	4,996	5,508	19.8	459.0	47.9
	19	181.60	5,953	1,483	21.7	123.6	166.1
	20	255.86	10,088	1,013	11.9	33.8	112.4
	21	216.77	6,921	1,535	19.7	34.1	43.6
	22	317.01	10,962	5,213	22.6	868.8	42.0
	23	177.45	3,246	572	18.8	143.0	29.9
	24	304.96	7,690	4,878	26.7	424.1	27.9
	25	311.34	11,217	1,247	22.7	831.1	2.9
	26	210.87	5,731	1,124	26.5	1,124.0	6.0
	27	260.86	9,933	79,641	20.1	1,276.3	6.7
1954–1955	28	309.12	7,371	2,368	17.8	263.1	52.1
	29	361.04	8,484	825	28.1	103.1	58.0
	30	248.12	6,880	2,111	13.8	263.9	110.9
	31	370.48	9,604	1,713	20.6	428.3	29.9
	32	727.55	23,760	1,859	29.1	619.7	6.8
	33	250.05	4,949	2,107	21.6	21.1	56.6
	34	267.58	5,669	4,583	18.6	305.6	73.5
	35	239.52	6,020	1,679	26.1	559.7	14.2
	36	482.09	16,340	1,128	18.3	20.5	225.4
	37	427.75	9,224	1,281	28.2	256.1	28.0
	38	121.03	1,454	1,283	20.0	1,282.7	26.3
	39	296.22	6,062	5,820	18.8	323.3	40.1
	40	544.23	19,830	2,048	21.4	113.8	157.5

(Continued on next page)

		X_1	X_4	X_5	X_6	X_7	X_8
1954-	41	390.14	9,185	2,749	28.2	687.3	11.1
1955	42	249.93	4,379	612	17.3	122.3	34.6
	43	332.29	8,540	5,547	23.0	554.7	18.8
	44	231.37	3,763	2,433	16.7	128.1	40.4
	45	275.94	4,860	7,464	19.1	622.0	47.9
	46	271.63	6,058	3,359	15.3	279.9	166.1
	47	319.19	8,239	1,855	16.7	61.8	112.4
	48	310.49	7,629	1,960	19.7	43.6	43.6
	49	359.11	9,765	6,970	23.5	1,161.6	42.0
	50	248.92	3,216	674	22.7	168.5	29.9
	51	328.06	7,093	6,210	25.4	540.0	27.9
	52	427.35	12,432	1,235	21.8	823.1	2.9
	53	271.81	6,042	1,193	25.0	1,192.7	6.0
	54	309.00	15,802	84,167	18.6	1,348.8	6.7

Source: St. Louis County Superintendent of Schools, *Annual Reports* (Clayton, Missouri, 1952 and 1955).

pupil in ADA, the following other independent variables are included in the analysis:

X_5 = number of pupils in ADA in public primary and secondary schools,[5]

X_6 = public high school pupils as a per cent of total pupils in ADA,

X_7 = number of public school pupils in ADA per square mile,

X_8 = per cent increase in public school pupils in ADA, 1951–1956.

These independent variables were correlated with X_1, annual total current expenditures plus debt service for public primary and secondary education per pupil in ADA.[6]

In Table 2, expenditure, assessed valuation, pupil number, high school pupils, density and growth data are given for the 27 school districts in the St. Louis City-County area with both primary and secondary schools in 1951–1952 and 1954–1955, respectively. This area has a population of about 1.5 million.

The multiple regression equation was found to have the following values for the years 1951–1952:[7]

$$X_1 = 49.714 + \underset{(.00122)}{.0177X_4} + \underset{(.00478)}{.0108X_5} - \underset{(.0000000583)}{.000000137X_5^2} + \underset{(1.275)}{3.633X_6} - \underset{(.0253)}{.0166X_7} - \underset{(.212)}{.558X_8}. \tag{4.1}$$

[5] Number of pupils was chosen as an independent variable to reflect the scale of operation of school districts. No similar national data are available going back to 1900.

[6] Since these are cross section data and the length of the school year in the St. Louis City-County area was about the same in all school districts in one period, it was possible to use annual in place of daily expenditure data.

[7] Consistent with experience in the private sector of the economy, it was assumed that the pupil number-expenditure function might be curvilinear.

The multiple regression equation was found to have the following values for the year 1954–1955:

$$X_1 = 92.859 + \underset{(.00120)}{\underline{.0229}}X_4 + \underset{(.00266)}{.00416}X_5 - \underset{(.0000000304)}{\underline{.0000000701}}X_5^2 + \underset{(1.584)}{\underline{3.371}}X_6 - \underset{(.0153)}{\underline{.0442}}X_7 - \underset{(.128)}{.259}X_8 . \tag{4.2}$$

Again, the figures in brackets are standard errors of the net regression coefficients. Under the circumstances, they are statistically significant at a probability level of .05 in case the ratio between coefficient and standard error is larger than 2.09, if we neglect the sign. Significant coefficients are underlined. The coefficient of determination adjusted for degrees of freedom lost ($\bar{R}^2_{1.4558678}$) for the two periods is .891 and .959, respectively. It is highly significant at a probability level of .05.

The Fabricant and Brazer cross section studies pertained to a single time period. The present study is of two successive time periods and makes possible comparisons over time. The net regression coefficient relating expenditures to assessed valuation of real property increased from .0177 in 1951–1952 to .0229 in 1954–1955. It was a statistically significant increase at a probability level of .05.

The average quasi-income elasticity of education was found to be + .594 for 1951–1952 and + .593 for 1954–1955. No significant change in the average quasi-income elasticity had taken place.[8]

Further comparisons are revealing. The 1951–1952 and 1954–1955 average quasi-income elasticities of the five major public education expenditure categories lend themselves to comparisons. Thus, the St. Louis City-County elasticity was + .29 for fixed charges, + .42 for instruction, + .50 for general control, + .55 for plant operation and maintenance and + 1.15 for auxiliary services. It is interesting to note that, with but one exception, the income elasticity for instruction is lowest. Fixed charges are the exception, where a low income elasticity is not surprising. By definition, these charges vary little with income. This finding points to the fact that even as more funds become available for public schools, there is no assurance that they will be channelled into those activities where they are needed most.

Finally, the quasi-income elasticity of education can be compared with that for fire protection and police protection in the St. Louis

[8] The increase in the magnitude of the net regression coefficient was offset by a decrease in the ratio between per pupil assessed valuation of real property and per pupil expenditures.

City-Country area during the same time period. The former was + 10.1 and the latter + .98.

Thus, no matter which of the two concepts is used, there can be little doubt that the income elasticity of public education is quite low. Especially low appears to be the income elasticity of instruction, which gives particular cause for concern. It is low in comparison to income elasticities of other public services and, in particular, of such consumer amenities as air-conditioning, automobiles, etc. It is also low compared to what it must be if public education in the United States is to be improved.

5. CONCLUSIONS

These empirical findings of relatively low income elasticity for public education pertain to a wealthy country with a federated political and tax system. So far the central government has contributed virtually nothing towards defraying the costs of education. It would be revealing to know more about the income elasticity of education in other countries, and whether countries with a federated system tend to have lower elasticities than those with a centralized one.

Recent universal concern with economic growth has once more focussed attention on the pivotal role played by education. Thus, great interest is expressed in getting people to spend a larger proportion of their incomes on education. To bring about positive changes appears easier in a centralized political and tax system than in a federated one. It is of interest to consider what countries with federated systems can do to raise the income elasticity of education.

Two major roads are open—raising the rates of existing proportional taxes and increasing the share of progressive taxes as well as their progressivity. In many countries proportional taxes, especially property and sales taxes, are the main sources of income of noncentral governments, and in some cases the electorate votes on tax rates. Under those circumstances, which are typical of the United States, enlightened and persuasive leadership might be needed to affect people's fundamental attitude toward public education and induce them to vote higher tax rates.

The second road can be travelled in two different ways. In addition to persuasion, fiscal incentives can be used. Most central governments rely more on progressive taxes than do the other governments. Thus, by providing macthing central funds the percentage of revenue raised through progressive taxation can be increased. The matching funds should not induce relaxation of local tax effort. Instead they should

increase the funds available for public education and assure judicious use.

With this in mind, matching central funds would best be tied to specific performance criteria in the form of eligibility requirements. Two important ones are minimum tax effort and a state- or province-wide floor below which educational opportunity is not allowed to fall. Thus, only jurisdictions that raise a given per cent of personal income for education should be eligible. In addition, a certain minimum education budget per child needs to be underwritten by each state or province. Fiscal incentives could also take other forms. For example, the central government could credit other governments with specified amounts in case they raise additional funds in the form of progressive taxes.

If the income elasticity for instruction is much lower than that for such other education expenditures as administration, plant operation and maintenance, auxiliary service and plant construction, "block" grants, i.e., grants for an unassigned general purpose in place of grants for specific purposes, will tend to be wasteful and not too effective in raising the quality of public education.

Much work remains to be done before a better understanding can be gained on how individual preferences are revealed through the voting mechanism and lead to social choice between private and public use of resources, in particular education. In the meantime, empirical income elasticities can help monitor a country's commitment to quality public education.

Washington University, U.S.A.

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A TEST OF THE CONSTANCY OF INPUT-OUTPUT COEFFICIENTS AMONG COUNTRIES*

BY TSUNEHIKO WATANABE[1]

1. INTRODUCTION

DURING THE 1950'S, a considerable number of interindustry tables and associated publications became available. Comparisons of these results by Chenery and Watanabe [3], Cao-Pinna [1], and Balboa [5] have led to the conclusion that a considerable degree of uniformity exists in the interindustry structure of production, despite marked differences in factor endowments, income levels, etc.

To extend these previous studies in a more precise way is the principal purpose of the present investigation. Specifically, the present paper tries to provide some empirical evidence as to the constancy of input-output coefficients using cross-country data from fifteen countries, and to indicate plausible interpretations of the results. The finding of fairly constant coefficients by many industries may be useful in development theories and planning methods.

2. BASIC DATA

The basic materials used here are input-output tables from fifteen countries, which include five countries with income per capita less than about $200 (India, Peru, Spain, Yugoslavia, and Japan), five countries with income per capita varying between $200 and $600 (Colombia, Italy, Argentina, Puerto Rico, and the Netherlands), and five countries with income per capita over $600 (Norway, U.K., Canada, Australia, and the U.S.).[2]

In the basic input-output tables, there were three important difficulties that might make the data not exactly comparable, i.e., (i) differences in the valuation of production, (ii) limitation on the size of the production matrix, and (iii) differences in the treatment of imports. It was almost impossible to make allowances for different methods of valuation except in the case of Italy. Therefore, conver-

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[2] Detailed information on input-output tables in these countries is contained in [2(Ch. 7)].

sion of purchaser's price into producer's price was undertaken for Italy only; whereas all others were left unchanged.[3] Most imports were charged to that industry which produces the same kind of commodity domestically; i.e., most imports were assumed to be competitive. The individual input-output tables were rearranged to follow a uniform classification, as in Table 1. The classification used contains twelve manufacturing industries, two primary industries, one service industry, and construction and electricity.[4] It was, however, easy to identify exogenous sectors, i.e., value added and final demand, in this way. Therefore, the basic framework of the input-output account reconsolidated here has the following accounting identity:

(Intermediate demand for seventeen industries)
+ (Intermediate demand for sector residual) + (Final demand)
≡ (Intermediate purchases of seventeen industries)
+ (Intermediate purchases of sector residual) + (Value added)
≡ (Output)

TABLE 1

INDUSTRY CLASSIFICATION

	U. N. international standard industrial classification
1. Agriculture, forestry, fishing	01, 02, 03, 04
2. Mining	11, 12, 13, 14, 19
3. Food manufacturing	20, 21, 22
4. Clothing	24
5. Textile manufacturing	23
6. Lumber and wood products	25, 26
7. Paper and printing	27, 28
8. Leather products	29
9. Rubber products	30
10. Chemicals	31
11. Petroleum and coal products	32
12. Nonmetallic mineral products	33
13. Metal products	34, 35
14. Machinery and transportation equipment	36, 37, 38
15. Construction	40
16. Electricity, gas, water service	51, 52
17. Tertiary	61, 64, 71, 73, 81, 83, 90
18. Residual	39

[3] Among fifteen countries, nine were valued at producer's price.

[4] Even at this level of aggregation, some countries were hard to fit in. In those cases, further aggregations of industries were made.

The input-output coefficients for each of the fifteen production matrices were calculated as the ratio of the intermediate flow in each cell to the corresponding total output.

3. THEORETICAL FRAMEWORK

There are two suggested hypotheses as to the constancy of input-output coefficients. They can be interpreted either as technological parameters that are defined in physical terms (Leontief [8]), or as constant in value terms (Klein [7]).

The Leontief production function is defined as a linear production function with fixed proportions of factor inputs. The inputs of the jth sector are denoted by $X_{ij}, j = 1, 2, \cdots, n$. The ratios

$$A_{ij} = \frac{X_{ij}}{X_j}$$

are defined as constant technical coefficients, where X_j = output of the jth industry and X_{ij} = output of industry i used as input in j. From a practical standpoint, however, the coefficients A_{ij} must be defined as ratios of values, since there is hardly any sector or industry which produces only one output. Klein's interpretation of Leontief's system meets this consideration. His theoretical model, as well as its empirical counterpart, proposes the constancy of the following ratio of values instead,

$$\bar{A}_{ij} = \frac{P_i X_{ij}}{P_j X_j},$$

where P's are prices of each output, and X_{ij} and X_j are the same as in the Leontief case. Assuming a perfectly competitive market economy and a treatment of profit as a payment to the capital factor, the marginal productivity doctrine of factor rewards leads this constancy of the input-output coefficients in value terms to the Cobb-Douglas production function with homogeneity of the first degree [7], [9]. The Cobb-Douglas is not the only admissible production function for the Klein model in general, but it is the only function which has an exact correspondence to the present assumptions. Conversely, if the production function is the Cobb-Douglas, the constancy of the in-put-output coefficients in value terms is obtained by the same assumptions.

In testing the plausibility of these two interpretations by using cross-country data, we have the limitation that no data with respect to prices are available, which means we cannot use quantities in terms of physical units in our test. Therefore, some theoretical qualification

is needed before statistical analysis can begin.

Following the neoclassical model, we shall write the production function as

$$X = f(X_1, \cdots, X_n), \tag{1}$$

where $X_1, \cdots, X_n$ are factors of production used in producing the output X. This function is assumed to satisfy all the neoclassical conditions. Given the prices of factors of production, $P_1, \cdots, P_n$, and of output P, the principle of profit maximization under a competitive market economy gives the following marginal conditions:

$$\frac{P_i}{P} = \frac{\partial f}{\partial X_i}, \qquad i = 1, \cdots, n. \tag{2}$$

Since our statistical information is limited to the product of the price and the quantity, i.e., PX and P_iX_i for all i, equation (2) is specified to give the following relations in order to derive the production function:

$$P_iX_i = c_i(PX), \qquad i = 1, \cdots, n, \tag{3}$$

where c_i is constant.[5] This suggests the Cobb-Douglas type of production function as a parametric form for (1).

There are several ways to evaluate the validity of equation (3) statistically. In the present investigation, we assume that a general linear relation exists between $(P_iX_i)^k$ and $(PX)^k$ or transformations of these variables. That is

$$\log (P_iX_i)^k = \log \beta_{0i} + \beta_{1i} \log (PX)^k + \varepsilon^k \tag{4}$$

was set up as the statistical equation to be tested using cross-country data, where ε is a random disturbance and k represents countries.[6]

The test criterion applied to equation (4) is that $\beta_{1i} = 1$ for all $i = 1, \cdots, n$.[7] If $\beta_{1i} = 1$ for all i, the estimates of β_{0i}, i.e., $\hat{\beta}_{0i}$, can

[5] From the basic tables, it is possible to distinguish the following variables:

$(P_jK_j)^k$ = the value of output in the jth industry of the kth country,

$(P_iX_{ij})^k$ = the value of the interindustry flow to the jth industry from the ith industry of the kth country,

$a_{ij}^k = (P_iX_{ij})^k/(P_jX_j)^k$ = the input-output coefficients,

where i and j represent industries 1 through 17, and k represents the countries from 1 to 15 as stated above. For the sake of convenience, the suffix j was omitted from the notation in the text.

[6] If we formulate the following statistical equation as an alternative model to equation (4),

$$(P_iX_i)^k = \alpha_{0i} + \alpha_{1i}(PX)^k + \xi^k,$$

the test criterion is that $\alpha_{0i} = 0$ for all i. In the following computations, this type of statistical equation was tested in several cases for the purpose of confirmation.

[7] Furthermore, this condition is needed to maintain the invariance of the results with respect to selection of exchange rates.

be used to estimate c_i. Furthermore, if $\Sigma\beta_{0i} = 1$, we can say that the production function is the Cobb-Douglas with homogeneity of the first degree. Then, the constancy of the input-output coefficients can be given an empirical examination only under Klein's interpretation, as mentioned above.[8]

The lack of sufficient variation of the relative price ratio (P_i/P) will give a significant limitation to the above interpretation. If the variation of (P_i/P) is not sufficiently large among countries concerned, which implies that our observations are concentrated in the local range of an equilibrium point on the production function (1), then different production functions, including the Leontief function, have to be considered.

4. STATISTICAL RESULTS

The first step in the statistical investigation of equation (4) is to specify the number of inputs to be tested in each industry. There are seventeen intermediate inputs and two primary factors, i.e., capital and labor. The actual procedure used here, under the restriction of the availability and the reliability of data, was as follows: (i) To determine several principal inputs to each industry, (ii) to prepare a coefficient of value added, which, due to the lack of independent information, must be a combination of the coefficients of capital and labor cost, (iii) to determine the residual input coefficient which corresponds to the value of unity minus the sum of the principal coefficients and value added ratio. The principal input coefficients were determined from the relative size of each coefficient in each industry and also from common technological knowledge. Table 2 gives the names of principal inputs which were selected for each of fifteen industries.[9]

This procedure resulted in the selection of three, four or five inputs in each industry.[10] For these selected inputs and the corresponding

[8] In order to test the Leontief production function with the same type of procedure as (4), we have to set up a statistical relation,

$$\log(P_iX_i) = \delta_0 + \delta_1 \log(PX) + \delta_2 \log\left(\frac{P_i}{P}\right) + \eta ,$$

where the criterion should be $\delta_1 = \delta_2 = 1$. Since price data are needed to test this relation, there is no direct way to examine the Leontief function in the present situation.

[9] Petroleum and coal products and tertiary sectors were omitted due to the significant lack of comparability in classification.

[10] The selection of inputs gave the following composition: Among 34 principal input coefficients, about half of them were composed of agriculture, chemicals and energy, and also leather products, paper and printing. Rubber products and construction did not occur as separate inputs.

TABLE 2

ESTIMATES OF PRODUCTION FUNCTIONS[a]

	Inputs	$\hat{\beta}_{i4}$	$S_{\hat{\beta}_{i4}}$	$\log\hat{\beta}_{04}$	$S_{\log\hat{\beta}_{04}}$	$\bar{R}^2$
	1 Agriculture ($N = 13$)					
3	Food	.7460	.248	− .8575	.872	.45
10	Chemicals	.9947	.313	−1.6248	1.110	.48
	Others	1.1155	.137	−1.0416	.482	.86
	Primary factors	.9986	.012	− .1699	.041	.99
	2 Mining ($N = 13$)					
13,14	Metal products and machinery	1.2270	.238	−1.2107	.640	.71
11,16	Energy	1.0218	.103	− .6148	.276	.90
	Others	1.0689	.178	− .9878	.150	.77
	Primary factors	.9572	.053	− .0439	.142	.97
	3 Food ($N = 14$)					
1	Agriculture	1.0700	.069	− .7162	.231	.95
17	Service, trade, and transport	1.0923	.059	−1.0843	.198	.96
	Others	.8910	.160	− .4235	.534	.72
	Primary factors	.9780	.064	− .5150	.212	.96
	4 Clothing ($N = 10$)					
5	Textiles	.9985	.055	− .4558	.153	.98
10	Chemicals	1.2149	.224	−1.8240	.620	.89
	Others	1.0978	.103	− .9047	.286	.93
	Primary factors	.9337	.054	− .2599	.128	.97
	5 Textiles ($N = 10$)					
1	Agriculture	1.0158	.065	− .6464	.189	.97
10	Chemicals	1.1874	.094	−2.0368	.271	.95
	Others	.9937	.058	− .5019	.169	.97
	Primary factors	.9770	.040	− .3530	.115	.99
	6 Wood products ($N = 13$)					
1	Agriculture	.9334	.037	− .3740	.091	.98
11.16	Energy	.9637	.064	− .9644	.156	.96
	Others	1.0498	.054	− .6757	.131	.97
	Primary factors	.9996	.044	− .4128	.107	.98
	7 Paper and printing ($N = 13$)					
11,16	Energy	1.0066	.113	− .8057	.295	.88
17	Service, trade, and transport	.9262	.047	− .7064	.127	.97
	Others	1.0026	.043	− .4396	.113	.98
	Primary factors	1.0114	.026	− .3664	.068	.99

[a] The regression formula used is

$$\log(P_i X_i) = \log \beta_{04} + \beta_{i4} \log(PX) + \varepsilon .$$

$\bar{R}^2$ represents the coefficient of determination, adjusted for degrees of freedom.

(*Continued on next page*)

	Inputs	$\hat{\beta}_{1i}$	$S_{\hat{\beta}_{1i}}$	$\log\hat{\beta}_{0i}$	$S_{\log\hat{\beta}_{0i}}$	$\bar{R}^2$
	8 Leather products ($N = 11$)					
1	Agriculture	.8394	.062	− .3878	.144	.95
10	Chemicals	.8165	.105	−1.0241	.244	.84
17	Service, trade, and transport	1.0440	.094	−1.1707	.217	.92
	Others	1.1993	.111	−1.0334	.256	.91
	Primary factors	1.0477	.050	− .5860	.115	.98
	9 Rubber products ($N = 9$)					
1	Agriculture	.9266	.083	− .6272	.152	.95
4,5	Clothing and textiles	1.0552	.058	− .9612	.107	.98
10	Chemicals	1.1376	.103	−1.4322	.191	.95
	Others	1.2275	.082	−1.1541	.150	.97
	Primary factors	.8702	.071	− .2101	.130	.96
	10 Chemicals ($N = 13$)					
1	Agriculture	.9762	.093	−1.2727	.256	.91
2	Mining	1.1210	.131	− .8134	.358	.87
11,16	Energy	.9135	.124	−1.2884	.338	.83
	Others	1.0338	.024	− .4210	.065	.99
	Primary factors	.9592	.024	− .2976	.065	.99
	12 Nonmetallic mineral products ($N = 13$)					
2	Mining	.9408	.091	− .7705	.210	.91
10	Chemicals	1.0369	.099	−1.7201	.231	.91
11,16	Energy	.8798	.137	−1.1158	.318	.80
	Others	1.0944	.080	− .8553	.185	.95
	Primary factors	.9506	.064	− .1565	.149	.95
	13 Metal products ($N = 12$)					
2	Mining	.9996	.034	− .8650	.103	.99
11,16	Energy	1.0691	.104	−1.5015	.312	.91
	Others	1.0616	.047	− .5751	.140	.98
	Primary factors	.9560	.061	− .2855	.183	.96
	14 Machinery ($N = 12$)					
10	Chemicals	.9739	.052	− .7409	.160	.97
13	Metal products	1.0039	.071	− .7801	.219	.95
	Others	1.0750	.045	− .7165	.138	.98
	Primary factors	.9508	.033	− .2043	.101	.99
	15 Construction ($N = 10$)					
6	Wood products	1.1455	.136	−1.6729	.447	.90
12	Nonmetallic mineral products	.8031	.089	− .3064	.186	.89
17	Metal products	.9983	.092	− .9611	.301	.94
	Others	1.0619	.108	− .8108	.354	.92
	Primary factors	.9920	.046	− .3362	.151	.98
	16 Electricity ($N = 13$)					
2, 11,16	Mining and energy	1.1459	.208	− .0624	.502	.74
	Others	1.0672	.268	− .2048	.649	.59
	Primary factors	.9776	.033	− .1783	.079	.99

output, equation (4) was fitted by the method of least squares.

The outcome of this procedure is summarized in Table 2, which gives the estimates of β_{1i} and of $\log \beta_{0i}$, the number of observations, and the coefficient of determination, $\bar{R}^2$.

In testing the hypothesis, $\beta_{1i} = 1$ for all i, the statistical significance of the departure of $\hat{\beta}_{1i}$ from unity must be based on the estimated standard error. For this purpose, if one of the estimates, $\hat{\beta}_{1i}$, in an industry did not satisfy the relation, $1 - 2S_{\hat{\beta}_{1i}} < \hat{\beta}_{1i} < 1 + 2S_{\hat{\beta}_{1i}}$, the hypothesis $\beta_{1i} = 1$ for this industry was rejected.[11] Following this procedure, three industries, leather products, rubber products, and construction, were found to have values significantly different from unity.

Also, since there is a choice of two regressions for the test of equation (3), using simple least-squares procedures, the same significance test must be applied to the regression coefficient, θ_{1i}, in the equation,

$$\log(PX)^k = \log \theta_{0i} + \theta_{1i} \log (P_iX_i)^k + \mu^k ,$$

where μ is a random disturbance. Applying this procedure, we omitted four industries—agriculture, mining, processed food, and construction—for the same reason as before.

As a result of these two kinds of testing, the null hypothesis that $\beta_{1i} = 1$ for all i was not rejected in seven manufacturing industries.[12] In these seven industries we may conclude that the Cobb-Douglas type production function is appropriate if we can assume sufficient variation in the relative price ratio, although the sample size may not be sufficiently large to feel perfectly sure about this conclusion.

Consideration was next given to the constancy of returns to scale, i.e., whether $\Sigma\hat{\beta}_{0i} = 1$ or not. In this case, there are two estimates of β_{0i}. The first estimate will be derived from equation (4), and the second estimate will be given by the following equation:

$$\log (P_iX_i)^k = \log \beta^*_{0i} + \log (PX)^k + \omega^k , \tag{5}$$

where the regression coefficient of $\log(PX)^k$ is assumed to be equal to one; i.e., the restriction $\beta_{1i} = 1$ is imposed a priori. Since the errors in equation (5) are assumed to be additive for logarithms, the estimate of β^*_{0i} can be computed as the ratio of geometric means of original variables. Estimated production functions for seven industries

[11] This criterion corresponds to the usual t-test with significance level at 5 per cent with sample size 10 to 14.

[12] It may be difficult to make a definite statement in regard to the above discussion. Although we are able to reject the null hypothesis, the acceptance of this hypothesis is subject to errors of the second kind.

are summarized in Table 3, where a corresponds to the estimate of equation (4) and b to the estimate of equation (5).

TABLE 3

ESTIMATES OF PRODUCTION FUNCTIONS

Clothing	a	$X_4 = A_4 X_3^{.350} X_{10}^{.092} X_0^{.125} X_{PF}^{.550}$	$\sum\hat{\beta}_{0i} = 1.027$
	b	$X_4 = A_4^* X_3^{.347} X_{10}^{.006} X_0^{.228} X_{PF}^{.365}$	$\sum\hat{\beta}_{0i}^* = .946$
Textiles	a	$X_5 = A_5 X_1^{.226} X_{10}^{.009} X_0^{.315} X_{PF}^{.444}$	$\sum\hat{\beta}_{0i} = .994$
	b	$X_5 = A_5^* X_1^{.230} X_{10}^{.031} X_0^{.383} X_{PF}^{.353}$	$\sum\hat{\beta}_{0i}^* = .966$
Wood products	a	$X_6 = A_6 X_1^{.423} X_{11+16}^{.011} X_0^{.311} X_{PF}^{.387}$	$\sum\hat{\beta}_{0i} = 1.032$
	b	$X_6 = A_6^* X_1^{.299} X_{11+16}^{.059} X_0^{.273} X_{PF}^{.388}$	$\sum\hat{\beta}_{0i}^* = 1.047$
Paper and printing	a	$X_7 = A_7 X_{11+16}^{.156} X_{17}^{.197} X_0^{.343} X_{PF}^{.430}$	$\sum\hat{\beta}_{0i} = 1.146$
	b	$X_7 = A_7^* X_{11+16}^{.162} X_{17}^{.129} X_0^{.369} X_{PF}^{.459}$	$\sum\hat{\beta}_{0i}^* = 1.119$
Chemicals	a	$X_{10} = A_{10} X_1^{.053} X_2^{.015} X_{11+16}^{.053} X_0^{.379} X_{PF}^{.504}$	$\sum\hat{\beta}_{0i} = 1.003$
	b	$X_{10} = A_{10}^* X_1^{.046} X_2^{.032} X_{11+16}^{.031} X_0^{.466} X_{PF}^{.393}$	$\sum\hat{\beta}_{0i}^* = .969$
Metal products	a	$X_{13} = A_{13} X_3^{.137} X_{11+16}^{.033} X_0^{.366} X_{PF}^{.518}$	$\sum\hat{\beta}_{0i} = .953$
	b	$X_{13} = A_{13}^* X_3^{.136} X_{11+16}^{.047} X_0^{.403} X_{PF}^{.393}$	$\sum\hat{\beta}_{0i}^* = .971$
Machinery	a	$X_{14} = A_{14} X_{10}^{.017} X_{13}^{.182} X_0^{.193} X_{PF}^{.625}$	$\sum\hat{\beta}_{0i} = 1.015$
	b	$X_{14} = A_{14}^* X_{10}^{.139} X_{13}^{.187} X_0^{.333} X_{PF}^{.445}$	$\sum\hat{\beta}_{0i}^* = 1.094$

Note: X_0 represents all other inputs in each industry, and X_{PF} shows combined inputs of capital and labor in each industry.

It may be noted here that complete estimation of a production function, which includes estimates of scale constants A_i, is possible only if price data are available. From Table 3, we would accept the hypothesis that $\Sigma\beta_{0i}$ or $\Sigma\beta_{0i}^*$ is equal to one. Therefore, in these seven industries, the Cobb-Douglas production function with homogeneity of degree one can be assumed and Klein's interpretation of the constancy of the input-output coefficients under the conditions stated in the previous section would be correct, assuming sufficient variation in prices.

5. A SUPPLEMENTARY TEST

So far, our discussions have been restricted to the principal coefficients. In this section we should like to extend our analysis to the whole set of coefficients. As mentioned earlier, there was not enough uniformity in either the industry classification or the valuation of production to make a complete analysis. The comparisons of the whole set of coefficients which we can do here are only supplementary. For this purpose a pair of industries in two countries was compared by the Wilcoxon test [10].[13] In every case, the Japanese input coefficients

[13] There are two reasons for using the Wilcoxon test: (i) The incompleteness in the

TABLE 4

COMPARISON OF MAJOR SECTORS

Industry \ Country	Argentina	Columbia	Europe	India	Italy	Netherlands	Peru	Puerto Rico	Spain	Yugoslavia	Relative frequency of r
Agriculture	a	*	a	a	r	a	a	a	a	r	2/9
Mining	r	*	a	a	a	a	a	*	a	a	1/8
Food manufacturing	a	a	a	a	a	a	a	a	a	a	0/10
Clothing	a	a			a	a	a				0/5
Clothing and textiles			a	a				r	a	a	1/5
Textile manufacturing	a	a			a	a	a				0/5
Chemicals	a	a	a	a	a	a	a	a	a	a	1/10
Nonmetallic mineral products	a	r	a	a	a	a	a	r	a	a	2/10
Metal products	a	*	a	a	a	*	a	a	a	a	0/8
Machinery	r	*	a	a	a	*	a	*	a	a	1/7
Electricity	a	*	a	a	r	a	a	a	a	a	1/9
Relative frequency of r	2/10	1/5	0/9	0/9	2/10	0/8	0/10	2/7	0/9	1/9	8/86

Note: a means the null hypothesis was accepted.
r means the hypothesis was rejected.
* means the data were not available.

were taken as standards of comparison. The null hypothesis to be tested was that a set of input-output coefficients in a specified industry did not differ between Japan and one of the other countries. That is, if this null hypothesis is true, the directions and the relative magnitudes of differences between each pair of input coefficients will be almost equal.

The results are given in Table 4. Among 86 pairs of comparisons, which include all available underdeveloped countries and several developed countries, only eight pairs are significantly different at the significance level of 5 per cent.[14]

reconsolidation of the production matrices was significant, and (ii) the effect of changes in input-output coefficients could be mainly considered through their direction of change and the relative magnitudes, as a first-order approximation. This was pointed out by Evans [6].

[14] This result is also subject to errors of the second kind.

This might be interpreted as a reflection of constancy in the principal coefficients in the previous section, and also as a confirmation of results in previous studies.

6. CONCLUSION

Combining the results derived in sections 4 and 5, we are able to draw the following tentative conclusions: (i) The degree of constancy of the principal input-output coefficients and the value added ratio among different countries can be expected to be high. The whole set of input-output coefficients shows about the same constancy, but our statement in this regard is less reliable. (ii) In manufacturing industries, the constancy is much higher than in other sectors, and the homogeneity condition is satisfied in these industries. The application of the Cobb-Douglas production function in these industries is therefore appropriate, if prices vary sufficiently. Since there was no available information on price changes, the choice between Leontief's and Klein's hypothesis could not be made conclusively. (iii) A typical input-output matrix can be usefully applied to those economies which do not have their own input-output data, especially to the case of less developed countries.[15]

Stanford University, U.S.A.

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[15] This type of experiment will be reported in [4].

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NOTE ON CONVENIENT MATRIX NOTATIONS IN MULTIVARIATE STATISTICAL ANALYSIS AND IN THE THEORY OF LINEAR AGGREGATION*

By T. Kloek

1. INTRODUCTION

The introduction of matrix notation and the use of matrix algebra have led to considerable simplifications in the treatment of those fields which are essentially quantitative, linear, and multivariate. In linear econometrics, for instance, this statement is valid for both of its theoretical constituents, mathematical economics and statistical analysis. The purpose of this note is to give two new contributions to the simplifications mentioned above.

The first of these (to be discussed in Section 2) has to do with variances and covariances. Measuring variables as deviations from their arithmetic means is a common procedure in statistics. Until now, however, it has been insufficiently realized that there exists a matrix corresponding to this linear transformation, the use of which may be profitable in certain mathematical proofs. It can be defined as

$$N = I - (1/T)\iota\iota' , \tag{1.1}$$

N being a square matrix of order T (which is the number of observations in the problem at hand), and ι a column vector of T unit elements. In Section 2.1 we shall consider some of the properties of N. Of course, applications are possible in several fields of statistics, especially in multivariate analysis. However, it would be beyond the the scope of this note to reconsider these topics. We shall, therefore, confine ourselves to regression analysis (Section 2.2) because of the outstanding role it has always played in econometric methodology. Finally, N appears to have an interesting generalization, which is useful in certain regression problems (Section 2.3).

The other contribution has to do with Theil's approach to the aggregation problem (Section 3). Many readers of his monograph on this subject [4] have been impressed by his ability to handle four, six and even more subscripts or superscripts. But some readers have also wondered whether it would not have been possible to simplify things a little by the use of a convenient matrix notation. The typical ele-

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ment of a matrix essentially has two indices, a row index and a column index, but if a scalar symbol has three indices, one might replace it by a matrix with one index, etc. The number of indices may thus be reduced by two, which will appear to be of great help in some of the proofs. In this note we shall confine ourselves to two cases, viz., aggregation over one set of individuals (Section 3.1) and aggregation over several sets of individuals (Section 3.2).[1]

2. MEASURING VARIABLES AS DEVIATIONS FROM THEIR MEANS

2.1. *Some Properties of the Matrix N.* In this section we consider the matrix N, defined in (1.1). In order to simplify our formulas, we introduce the following symbols:

$$N_T = (1/T)N\,; \quad \iota_T = (1/T)\iota\,. \tag{2.1}$$

Hence (1.1) can be rewritten as

$$N = I - \iota\iota_T'\,. \tag{2.2}$$

Further, let us consider a column vector x consisting of T sample observations on some variable. Its arithmetic mean $\bar{x}$ may be written as

$$\bar{x} = \iota_T'x\,. \tag{2.3}$$

Premultiplying x by N, we get

$$Nx = x - \iota\iota_T'x = x - \iota\bar{x}\,, \tag{2.4}$$

and we may conclude that the result of this transformation is that we have measured the elements of x as deviations from their mean $\bar{x}$.

The matrix N has several interesting properties, of which the following are obvious:

$$N' = N\,, \tag{2.5}$$

$$N'N = N^2 = N\,, \tag{2.6}$$

$$N\iota = 0\,, \tag{2.7}$$

$$\operatorname{tr} N = T - 1\,. \tag{2.8}$$

In words: N is symmetric and idempotent; its rows (columns) have zero sums; and its trace is $T - 1$. It is a property of idempotent matrices that their characteristic roots are either zero or unity.[2] If we combine this with the well known theorem that the sum of all characteristic roots of a matrix is equal to its trace, we find that one

[1] See Theil [4 (Ch. 2 and Section 3.2)].

[2] See e.g. Gantmacher [3, (226)].

of the roots is zero, and all others are unity. It follows from (2.7) that the charactristic vector corresponding to the zero root is equal to ι, apart from normalization. Since the unit root is $(T-1)$-fold, the corresponding characteristic vectors can be chosen arbitrarily, provided they are orthogonal to ι and to each other[3]. In statistical terms, any set of $T-1$ uncorrelated variables with zero means, may be regarded, together with ι, as the set of all characteristic vectors of N. The following geometric interpretation is illustrative. The linear transformation corresponding to N transforms the points of a T-dimensional space such that the image of each point coincides with its projection on the $(T-1)$-dimensional hyperplane through the origin, perpendicular to ι.

2.2. *The Role of the Matrix N in Regression Analysis.* From (2.1), (2.4) and (2.6) it follows that we can use N (or N_r) to define variances and covariances in a simple way. For instance, if we have T observations on A variables written in the form of a $T \times A$ matrix

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1A} \\ \cdots & \cdots & \cdots \\ x_{T1} & \cdots & x_{TA} \end{bmatrix}, \tag{2.9}$$

we may write for the covariance matrix of these variables

$$V(X) = \frac{1}{T} X'NX = X'N_rX\,. \tag{2.10}$$

Let us now consider a linear regression equation[4]

$$y = X\beta + u\,, \tag{2.11}$$

where y is a column vector of T observations on the variable to be explained, X is a $T \times A$ matrix containing the values taken by A explanatory variables including a column ι corresponding to the constant term, β is a vector of A parameters, and u is a disturbance vector. Usually it is assumed that X is of rank A with nonstochastic elements and that the elements of u are stochastic with zero expectation and a scalar covariance matrix $\sigma^2 I$.

Since β and u are unknown, (2.11) may be estimated by

[3] The fact that the characteristic vectors all exist and can be made orthogonal to each other follows from the fact that N is real and symmetric. See, e.g., Zurmühl [6, (184, 185)].

[4] If we would have instead an equation which is part of a system of simultaneous equations, to be estimated by the method of two-stage least-squares, the argument is similar.

$$y = Xb + v \,. \tag{2.12}$$

It is well known that least-squares estimation (i.e., minimization of $v'v$) leads to the "normal equations"[5]

$$X'y = X'Xb \,, \tag{2.13}$$

from which the solution for b is

$$b = (X'X)^{-1}X'y \,, \tag{2.14}$$

and that the sampling distribution of b has the covariance matrix

$$V(b) = \sigma^2(X'X)^{-1} \,. \tag{2.15}$$

In order to show what happens when one wishes to work with variables which are taken as deviations from their means, it will prove convenient to partition X, β and b as follows:[6]

$$X = [Z \quad \iota] \,, \quad \beta = \begin{Bmatrix} \beta_z \\ \alpha \end{Bmatrix} \,, \quad b = \begin{Bmatrix} b_z \\ a \end{Bmatrix} \,, \tag{2.16}$$

α and a being scalars (the "constant terms"). Hence (2.14) is to be written as

$$\begin{Bmatrix} b_z \\ a \end{Bmatrix} = \begin{bmatrix} Z'Z & Z'\iota \\ \iota'Z & T \end{bmatrix}^{-1} \begin{bmatrix} Z'y \\ \iota'y \end{bmatrix} . \tag{2.17}$$

We also partition $(X'X)^{-1}$ as follows:

$$\begin{bmatrix} P & q \\ q' & r \end{bmatrix} = \begin{bmatrix} Z'Z & Z'\iota \\ \iota'Z & T \end{bmatrix}^{-1} . \tag{2.18}$$

P is a square and symmetric matrix of order $\Lambda - 1$; q is a column vector of $\Lambda - 1$ elements; q' is its transpose; and r is a scalar. Combining (2.17) and (2.18) we find

$$b_z = PZ'y + q\iota'y \,. \tag{2.19}$$

If we write the statement $(X'X)^{-1}X'X = I$ in a partitioned way, we find the conditions

$$PZ'Z + q\iota'Z = I \,, \tag{2.20}$$

$$PZ'\iota + qT = 0 \,, \tag{2.21}$$

from which P and q can be determined. Starting from (2.21), we

[5] Under the assumptions made and an additional multinormality assumption on the distribution of the components of u, the maximum-likelihood criterion implies the least-squares criterion.

[6] Usually (and also in Section 3) X is partitioned as $[\iota \quad Z]$, but in this context it is easier to take the reverse order.

can express q in terms of P and Z

$$q = -(1/T)PZ'\iota \,. \tag{2.22}$$

Substituting this in (2.20), we find

$$PZ'Z - (1/T)PZ'\iota\iota'Z = I \,, \tag{2.23}$$

which can be written as

$$PZ'NZ = I \,. \tag{2.24}$$

It follows that

$$P = (Z'NZ)^{-1} \,. \tag{2.25}$$

Combining (2.19), (2.22) and (2.25), we find

$$b_z = PZ'y - (1/T)PZ'\iota\iota'y = (Z'NZ)^{-1}Z'Ny \,. \tag{2.26}$$

Often variables are taken as deviations from their means at the beginning. This amounts to premultiplying (2.12) by N

$$Ny = NZb_z + N\iota a + Nv \,. \tag{2.27}$$

The second term on the right-hand side vanishes because of (2.7). The application of the least-squares procedure to (2.12) implies that v is orthogonal to all columns of X, and also to ι. Consequently, we may consider v as a characteristic vector of N corresponding to a unit root[7]

$$Nv = v \,, \tag{2.28}$$

and we may rewrite (2.27) in the form

$$Ny = NZb_z + v \,. \tag{2.29}$$

Minimization of $v'v$, according to (2.29), leads immediately to the expression after the second equality sign in (2.26).[8] Both approaches, therefore, lead to the same result. Given b_z computed according to the second procedure, it is possible to find a in an easy way from

$$a = (1/T)(\iota'y - \iota'Zb_z)_v \,. \tag{2.30}$$

This can be verified by writing

$$a = q'Z'y + r\iota'y \,, \tag{2.31}$$

which follows from (2.17) and (2.18); r can be determined from the condition

[7] Cf. Section 2.1.

[8] Sometimes the N transformation is applied only to the explanatory variables and not to the dependent variable. In that case we have, instead of (2.29), $y = NZb_z + v$. It is easily seen that this does not change our result.

$$q'Z'\iota + rT = 1 , \tag{2.32}$$

derived from (2.18):

$$r = (1/T)(1 - q'Z'\iota) = (1/T) + (1/T)^2\iota'Z(Z'NZ)^{-1}Z'\iota , \tag{2.33}$$

with q being given by (2.22) and (2.25).

The covariance matrix of b_x, $V(b_x)$, is the leading $(A - 1)$-sub-matrix of $V(b)$. By (2.15), (2.18) and (2.25), we find

$$V(b_x) = \sigma^2 P = \sigma^2(Z'NZ)^{-1} . \tag{2.34}$$

If we apply the deviation approach of (2.29), we again find that both methods lead to the same formula. For the variance of the constant term and the covariances of the constant term and the multiplicative coefficients we have, finally,

$$\text{var } a = \sigma^2 r = \sigma^2\{(1/T) + (1/T)^2\iota'Z(Z'NZ)^{-1}Z'\iota\} \tag{2.35}$$

$$\text{cov}(a, b_x) = \sigma^2 q = -\sigma^2(1/T)(Z'NZ)^{-1}Z'\iota . \tag{2.36}$$

The contents of most of our conclusions are well known but it is a rather complicated affair to prove them without making use of the N matrix.

2.3. *A Generalized Transformation.* The following generalization of the N transformation has some interesting applications. We now partition X and β in equation (2.11) in a somewhat different fashion

$$X = [X_1 \; X_2] ; \quad \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} , \tag{2.37}$$

such that (2.11) can be written as

$$y = X_1\beta_1 + X_2\beta_2 + u . \tag{2.38}$$

It is possible to write X_1 linearly in terms of X_2 and a residual matrix W, as follows:

$$X_1 = X_2(X_2'X_2)^{-1}X_2'X_1 + W , \tag{2.39}$$

which implies

$$W = \{I - X_2(X_2'X_2)^{-1}X_2'\}X_1 . \tag{2.40}$$

We now have a straightforward generalization of N in the form of

$$M_2 = I - X_2(X_2'X_2)^{-1}X_2' , \tag{2.41}$$

which is easily verified by replacing X_2 by ι. M_2 also appears to be symmetric and idempotent, while (2.7) and (2.8) are generalized by

$$M_2X_2 = 0 \tag{2.42}$$

$$\text{tr } M_2 = T - \Lambda_2 , \tag{2.43}$$

Λ_2 being the number of columns of X_2. There are Λ_2 zero roots and $T - \Lambda_2$ unit roots. The characteristic vectors corresponding to the former can be written as linear combinations of the columns of X_2, while those corresponding to the latter are orthogonal to the columns of X_2 (e.g., v).

Proceeding in the same way as in Section (2.2), it is easily proved that the least-squares estimator b_1 of β_1 can be written as

$$b_1 = (X_1'M_2X_1)^{-1}X_1'M_2y . \tag{2.44}$$

Again, the same result may be obtained by first transforming all variables by means of premultiplication by M_2

$$M_2y = M_2X_1b_1 + v \tag{2.45}$$

(making use of $M_2X_2 = 0$ and $M_2v = v$) and then estimating b_1 by the method of least squares.

The latter method is computationally simpler if one is not interested in b_2 (the constant term in the case of Section 2.2), while there is no difference in this respect in the opposite case.

An important consequence of this general method is that it does not make any difference for the results whether a trend or a seasonal pattern is included among the explanatory variables, or removed in advance from the other variables, provided this is done in a linear way.[9]

3. LINEAR AGGREGATION OF ECONOMIC RELATIONS

3.1. *Aggregation over One Set of Individuals.*[10] Since the only purpose of this note is to discuss some simplifications in notation, the ideas of Theil's aggregation theory will be explained very concisely. For an extensive treatment reference is made to his monograph [4], for a short survey to [5] and for an empirical application to Boot and De Wit [1].

We start by considering K individuals who have each their own linear behavior relation founded on microeconomic theory. We write these equations in the form of linear regression equations

$$y_k = X_k\beta_k + u_k \qquad (k = 1, \cdots, K) , \tag{3.1}$$

y_k being a column vector of T values taken by the endogenous variable, X_k being a $T \times \Lambda$ matrix of nonstochastic values taken by the exog-

[9] For the special case of the trend, this was proved by Frisch and Waugh [2].

[10] See Theil [4 (Ch. 2)].

enous variables,[11] β_k being a column vector consisting of A unknown microparameters, and u_k being a disturbance vector consisting of T random elements with zero expectation.

Further, an analogous macrorelationship is postulated

$$y = X\beta + u \,, \tag{3.2}$$

where y and X are aggregates[12], defined by

$$y = \Sigma y_k \,, \tag{3.3}$$

$$X = \Sigma X_k \,. \tag{3.4}$$

The summation in (3.3)–(3.4), as well as in the rest of this subsection, is extended over $k = 1, \cdots, K$; β is defined as the mathematical expectation of its least-squares estimator[13]

$$\beta = E\{(X'X)^{-1}X'y\} \,; \tag{3.5}$$

and u is defined implicitly.

In order that it be possible to express the macroparameters β in terms of the microparameters β_k, the so-called auxiliary equations are introduced

$$X_k = XB_k + V_k \qquad (k = 1, \cdots, K) \,, \tag{3.6}$$

B_k being computed by the method of least squares[14]

$$B_k = (X'X)^{-1}X'X_k \qquad (k = 1, \cdots, K) \,. \tag{3.7}$$

The relationships (3.6) are entirely formal; no direct economic meaning should be attached to them.

By substitution of (3.3), (3.1), and (3.7) in (3.5) we find[15]

$$\beta = E\{(X'X)^{-1}X'\Sigma(X_k\beta_k + u_k)\} = \Sigma B_k\beta_k \,, \tag{3.8}$$

and by substitution of (3.1), (3.6), and (3.8) in (3.3)

[11] In order to be in agreement with our definition (3.4), we take, if (3.1) contains a constant term, the first column of X_k not as ι (a column vector of T unit elements) but as $(1/K)\iota$, and the first element of β_k not as the constant term but as K times the constant term.

[12] Sum aggregates may be replaced by fixed-weight aggregates (i.e., index numbers) without loss of generality. See Theil [4 (12)].

[13] It is assumed that X has rank A.

[14] If the relationship (3.1) contains a constant term, it is easily seen that the first column of B_k is equal to the transpose of $(1/K)$ $(1\ 0 \cdots 0)$, and that the first column of V_k is a null vector (all k). [Hint: partition X_k and X such that $(1/K)\iota$ and ι are written explicitly.]

[15] The results (3.8) and (3.12) are equivalent to Theil's Theorem 1 [4 (13, 15)], while (3.10) and (3.13) are special cases of his formulas (9.10) and (9.13) [4 (185. 186)].

$$y = \Sigma(XB_k + V_k)\beta_k + \Sigma u_k = X\beta + \Sigma V_k\beta_k + \Sigma u_k . \tag{3.9}$$

Combining (3.2) and (3.9) we get

$$u = \Sigma V_k\beta_k + \Sigma u_k . \tag{3.10}$$

Further, it is possible to substitute (3.6) in (3.4), which leads to the identity

$$X = X\Sigma B_k + \Sigma V_k . \tag{3.11}$$

We conclude

$$\Sigma B_k = I \tag{3.12}$$

$$\Sigma V_k = 0 . \tag{3.13}$$

which are the weight rules in linear aggregation.

Finally, we consider the least-squares estimator of the vector of macroparameters β. Making use of (3.3), (3.1), (3.7), and (3.8) we find[16]

$$b = (X'X)^{-1}X'\Sigma X_k\beta_k + (X'X)^{-1}X'\Sigma u_k = \beta + (X'X)^{-1}X'\Sigma u_k ; \tag{3.14}$$

hence $(X'X)^{-1}X'\Sigma u_k$ is the implicit sampling error of the macroeconomic least-squares estimator.

3.2. *Aggregation over Several Sets of Individuals.*[17] Let us now consider a somewhat more general case. Suppose that there are K individuals or commodities and K (endogenous) microvariables corresponding to them. Suppose that these variables depend linearly on several other exogenous microvariables possibly belonging to other sets of individuals or commodities. Then we can write

$$y_k = \sum_{h=1}^{H} X_h\beta_{hk} + u_k \qquad (k = 1, \cdots, K) , \tag{3.15}$$

where H is the number of individuals or commodities corresponding to the macrovariables X.[18]

The postulated macrorelationship has again the form (3.2); y and β are again defined by (3.3) and (3.5), respectively. For X we now have

$$X = \sum_h X_h , \tag{3.16}$$

and for the auxiliary equations

$$X_h = XB_h + V_h , \tag{3.17}$$

[16] See Boot and De Wit [1 (10)].

[17] See Theil [4 (Sec. 3.2)].

[18] In fact, this number may be different for the different columns of X_h, but this can easily be remedied by putting zero vectors in the empty columns.

where

$$B_h = (X'X)^{-1}X'X_h \,. \tag{3.18}$$

Proceeding now as in Section 3.1, we find[19]

$$\beta = \sum_h B_h \sum_h \beta_{hh} \,, \tag{3.19}$$

$$u = \sum_h V_h \sum_h \beta_{hh} + \sum_h u_h \,, \tag{3.20}$$

$$\sum_h B_h = I \,, \tag{3.21}$$

$$\sum_h V_h = 0 \,, \tag{3.22}$$

while (3.14) is valid without change.

Econometric Institute of the Netherlands School of Economics, Netherlands.

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[19] The results (3.19) and (3.21) are equivalent to Theil's Theorem 2 [4 (30)]; (3.20) and (3.22) are again special cases of his formulas (9.10) and (9.13) [4 (185, 186)]

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DURABILITY OF CAPITAL AND RATE OF GROWTH OF NATIONAL PRODUCT*

BY LEIF JOHANSEN

1. INTRODUCTION

IN RECENT YEARS the relation between the observed rate of growth of national product and the investment quota has been a central theme in current economic debate in Norway. The investment quota has been higher than in most other countries, but the rate of growth of national product has not been "correspondingly" high. In 1959, the volume index for gross national product on a 1953 base was 119 for Norway; whereas the index for all OEEC countries combined was 128. Taking 1955 as a typical year, the gross investment quota was 31 percent for Norway, compared with 18 percent for all OEEC countries combined.

Now if one does not stick to the simplest type of growth models it is very easy to find possible explanations of the fact that a country has a high investment quota without having a high growth rate. For instance, the growth of the labor force should obviously be taken into account.[1]

The purpose of this paper is to examine one special argument which has been advanced in connection with the problem mentioned above. It has been argued that the average durability of capital in some of the most important Norwegian industries, such as hydroelectric power plants, ships etc., is very high. This should be a drawback in the attempt to achieve a high growth rate.

On the other hand, it has been argued that if the investment in these sorts of durable capital equipment yields a normal rate of interest, it should be as good as other investment. Alternatively, if investment does not promote growth of production to the same extent as in most other countries it must mean that investment in Norway is less profitable than investment in other countries.

In order to state the problem precisely one must clearly distinguish between gross national product and net national product, and between gross investment and net investment. In most international comparisons the gross concepts are used. Theoretically, however, it may be more interesting to consider the net concepts. We shall, in this

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[1] See, e.g., [4]. The model used there is a special case of a model presented in [3].

paper, examine the validity of the argument both as applied to gross and to net concepts.

Further, it is important to distinguish between the effects of durability on the asymptotic growth rate and the immediate effects of a switch-over from investment with a lower to investment with a higher durability.

In order to isolate the matter of the durability of capital from other aspects of the problem, I find it reasonable to compare cases in which the yields of capital as measured by internal rates of interest are the same while the durabilities are different.

Other authors have investigated the effect of the durability of capital on the growth rate, but on other assumptions. In [2], E. Domar considers the problem of finding optimum durability with respect to the growth rate of national product when the investment quota is kept constant, and there is a technical relationship between durability and the capital coefficient, i.e., when "greater capital longevity must be paid for."

In [1], H. Brems investigates the effects of technical progress which increases the durability of capital without affecting its productivity. However, none of these studies considers differences in the durability of capital which are compensated by differences in the productivity of capital in such a way that the internal rate of interest remains the same.

The model which we shall employ is a pure capital accumulation model. However, in the final section we shall offer some suggestions as to how the growth of the labor force could be introduced in the picture without altering the nature of the conclusions.

2. THE ASYMPTOTIC GROWTH RATE

We shall consider the problem for the asymptotic growth rate. We proceed by studying first a "gross model" and then a "net model". These models are similar to E. Domar's β-model and α-model, respectively, [2].

2a. *The gross model.* The model contains the following variables and coefficients:

X is the gross national product;
K is the total stock of capital;
k is the gross investment;
ξ is the productivity of capital;
β is the gross investment quota;
T is the durability of capital equipment.

We assume only one sort of product, with a constant price equal to unity, which can be used alternatively for consumption or accumulation.

Assuming that capital retains its full productivity over its useful life span, we have the following model:

$$k(t) = \beta X(t) , \tag{2.1}$$

$$X(t) = \xi K(t) , \tag{2.2}$$

$$K(t) = \int_{t-T}^{t} k(\tau)d\tau . \tag{2.3}$$

From this model one obtains

$$X(t) = \beta\xi\int_{t-T}^{t} X(\tau)d\tau ,$$

with a corresponding characteristic equation

$$\rho = \beta\xi(1 - e^{-\rho T}) . \tag{2.4}$$

For plausible values of β and ξ this yields a unique real solution for ρ, which is the asymptotic growth rate of the national product (and also for the other variables of the model); cf. the discussion in [3 (section 5)].

Our problem now is to investigate how ρ is affected by the value of T, keeping the internal rate of interest constant. We must first define this rate in terms of ξ and T.

Let the internal rate of interest be r. This rate should be defined such that the present value of the yield of one unit of capital over its life span should equal the price of this unit of capital, when r is used as the discount factor, i.e.,

$$1 = \int_0^T \xi e^{-r\tau}d\tau ,$$

or

$$r = \xi(1 - e^{-rT}) . \tag{2.5}$$

This relationship defines the changes in ξ which are necessary to compensate for changes in T such that the internal rate of return r remains constant.

Inserting the expression for ξ from (2.5) into (2.4), we obtain the relationship between ρ, r and T,

$$\frac{\rho}{1 - e^{-\rho T}} = \beta\frac{r}{1 - e^{-rT}} . \tag{2.6}$$

It is convenient to introduce the function

$$g(r, T) = \frac{r}{1 - e^{-rT}}, \tag{2.7}$$

and write (2.6) as

$$g(\rho, T) = \beta g(r, T). \tag{2.8}$$

This is the formal solution to our problem. We can keep r (and β) constant and see how durability T influences the rate of growth ρ.

The relationship (2.8) might be studied graphically, or it might be subjected to implicit differentiation. Rather than carry out such studies, I shall present a few numerical examples with plausible coefficients. These are set out (as percentages) in Table 1. In all cases the gross investment quota is kept at $\beta = 1/3$, which is an approximation to Norwegian conditions.

TABLE 1

THE ASYMPTOTIC (PERCENTAGE) RATE OF GROWTH OF GROSS NATIONAL PRODUCT FOR DIFFERENT VALUES OF DURABILITY OF CAPITAL EQUIPMENT (T) AND THE INTERNAL RATE OF RETURN TO CAPITAL (r).

(THE GROSS INVESTMENT QUOTA $\beta = 1/3$)

$100r$ \ T	10	20	30	40	50	100	∞
10	<0	<0	0.4	1.7	2.3	3.2	3.3
15	<0	0.5	2.9	4.0	4.5	5.0	5.0
20	<0	3.3	5.4	6.1	6.4	6.6	6.7
25	<0	5.7	7.4	8.0	8.2	8.3	8.3

In order that growth shall be positive in our model it is necessary that

$$\beta \xi T = \beta T g(r, T) > 1. \tag{2.9}$$

Cases for which this condition is not fulfilled are simply marked "<0" in Table 1.

As $T \to \infty$, it is seen that $g(r, T) \to r$. In this case $\rho \to \beta r$, which is indicated in the last column in Table 1. Thus, for a fixed gross investment quota and a given internal rate of return to capital, the asymptotic rate of growth of the national product increases with the durability of capital and approaches an upper limit as $T \to \infty$.

It is seen from the table that differences in T, which are not larger than one might expect to find when comparing different countries, may explain considerable differences in the rates of growth. This observation may be useful in connection with international comparisons which are carried out in terms of gross concepts.

The direction of the effect in this case is fairly obvious from the following observation. In the case of short-lived capital, the rate of depreciation will be large. A given *gross* investment quota will then imply a small *net* investment quota. In the case of long-lived capital, the same gross investment quota will imply a larger net investment quota. The purpose of Table 1 is to illustrate just how strongly this will affect the growth rate.

2b. *The net model.* Let capital be amortized according to the straight line method over its life span T, as is usual in national accounting. Net investment is then

$$I(t) = k(t) - \frac{1}{T} K(t) , \tag{2.10}$$

and net national product is

$$R(t) = X(t) - \frac{1}{T} K(t) . \tag{2.11}$$

We now assume net investment to be a fraction α of net national product, i.e.,

$$I(t) = \alpha R(t) . \tag{2.12}$$

Our "net model" then consists of equations (2.2–3), together with (2.10–12).

Instead of (2.4) we now obtain the characteristic equation

$$\rho = \left(\alpha\xi + \frac{1-\alpha}{T}\right)(1 - e^{-\rho T}) , \tag{2.13}$$

and instead of (2.8) we obtain

$$g(\rho, T) = \alpha g(r, T) + \frac{1-\alpha}{T} . \tag{2.14}$$

For any given internal rate of return r and net investment quota α, this now defines the relationship between the asymptotic growth rate ρ and the durability of capital T.

Table 2 gives numerical values for the net model similar to those in Table 1 for the gross model. We have chosen $\alpha = 0.20$, which is an approximation to Norwegian conditions. We see that within relevant ranges the durability of capital has a rather small influence on the growth rate in the setting of the model where the *net* investment quota is kept constant. We observe that there exists an optimum T with respect to the growth rate for each value of r, and this optimum

T is smaller the larger is r. Since the influence of T is small, however, this optimum value is not sharply located and is of little practical interest.

TABLE 2

THE ASYMPTOTIC (PERCENTAGE) RATE OF GROWTH OF NET NATIONAL PRODUCT FOR DIFFERENT VALUES OF DURABILITY OF CAPITAL EQUIPMENT (T) AND THE INTERNAL RATE OF RETURN TO CAPITAL (r).

(NET INVESTMENT QUOTA $\alpha = 0.20$)

$100r$ \ T	5	10	20	30	40	50	100	∞
10	2.2	2.2	2.5	2.5	2.6	2.6	2.6	2.0
15	3.3	3.6	3.8	4.0	4.0	4.0	3.7	3.0
20	4.6	4.9	5.2	5.4	5.3	5.2	4.8	4.0
25	5.7	6.2	6.6	6.6	6.4	6.3	5.8	5.0

3. THE IMMEDIATE EFFECTS OF A CHANGE IN DURABILITY

The previous section was concerned with the growth rate at which the economy will ultimately settle down if r, β or α, and T are kept constant. In the present section we shall consider the following problem: Suppose that r, β or α, and T have been kept constant for some period and that the economy has been growing regularly according to a growth rate ρ as determined in the previous section. Let these particular values of T and ρ be denoted by T_0 and ρ_0. From a certain point of time, however, all new capital constructed has a durability $T_1 > T_0$, while r is still the same as before.

The constancy of r while T changes from T_0 to T_1 where $T_1 > T_0$, implies that we have to distinguish between "new" and "old" capital in the production function (2.2) and apply a coefficient $\xi = \xi_0$ for old capital and a coefficient $\xi = \xi_1$ for new capital, where $\xi_1 < \xi_0$.

As in the previous section we consider in turn the gross model and the net model.

3a. *The gross model.* Let the point of time for the switch-over be $t = 0$. The relative growth rate for the gross national product X from $t = 0$, which we denote by ρ_x, is then

$$\rho_x = \frac{1}{X(t)} \frac{dX(t)}{dt} = \frac{\dot{X}(t)}{X(t)}, \quad \text{for } t = 0, \tag{3.1}$$

where the derivative is a "right hand derivative".

We now have

$$\dot{X}(0) = \xi_1 k(0) - \xi_0 k(-T_0). \tag{3.2}$$

The first term represents the increase in production due to the gross investment $k(0)$, which now takes the shape of capital with durability T_1 and productivity ξ_1. The second term represents the loss of production due to scrapping of capital which was produced T_0 periods earlier with productivity ξ_0.

Since the economy has been growing at a rate ρ_0 in the relevant past, we have

$$k(-T_0) = k(0)e^{-\rho_0 T_0}, \tag{3.3}$$

and since we have further that $k(0) = \beta X(0)$, we can write (3.2) as

$$\rho_x = \beta(\xi_1 - \xi_0 e^{-\rho_0 T_0}). \tag{3.4}$$

The problem now is to compare ρ_x with ρ_0.

The value of ρ_0 is determined by (2.4) for $T = T_0$. Combining this with (3.4), we obtain

$$\rho_x = \rho_0 - \beta(\xi_0 - \xi_1). \tag{3.5}$$

Expressing ξ_1 and ξ_0 in terms of r and T_1 or T_0 by means of (2.5), we can write this as

$$\rho_x = \rho_0 - \beta[g(r, T_0) - g(r, T_1)]. \tag{3.6}$$

Since $g(r, T)$ decreases with T, we have $\rho_x < \rho_0$ for $T_1 > T_0$.

Numerical evaluation on the basis of this formula shows that the immediate effect of a switch-over from one value of T to another which is, for instance, 10 years higher is generally not very large. It may amount to a fraction of one per cent. For example, with $\beta = 1/3$, $r = 0.15$, and a change in T from $T_0 = 20$ to $T_1 = 30$, the growth rate ρ drops from 0.5 to 0.3 per cent. With $r = 0.20$ in the same case, ρ drops only from 3.3 to 3.2 per cent. For larger values of r and T_0 the effect of such an increase in T is even smaller.

3b. *The net model.* In this case we keep the net investment quota α constant, and we concentrate attention upon the growth rate of net national product R. As in (3.1), we denote the growth rate of R immediately after the change in T by ρ_R, i.e.,

$$\rho_R = \frac{1}{R(t)}\frac{dR(t)}{dt} = \frac{\dot{R}(t)}{R(t)}, \quad \text{for } t = 0. \tag{3.7}$$

We now have

$$\dot{R}(0) = \dot{X}(0) - \frac{1}{T_1}k(0) + \frac{1}{T_0}k(-T_0), \tag{3.8}$$

where $\dot{X}(0)$ is given by (3.2).

Using (3.3) and equations (2.10–14), we obtain the following expression for ρ_R:

$$(3.9) \qquad \rho_R = \rho_0 + \left[\alpha g(r, T_0) + \frac{1-\alpha}{T_0}\right]\left|\frac{g(r, T_1) - \frac{1}{T_1}}{g(r, T_0) - \frac{1}{T_0}} - 1\right|.$$

The expression $g(r, T) - 1/T$ increases with T when $r > 0$. We therefore have $\rho_R > \rho_0$ for $T_1 > T_0$. This means that an increase in the durability of capital leads immediately to an increase in the rate of growth of net national product when the internal rate of return to capital is the same and the net investment quota is kept constant. The advantage, from this point of view, of a slower increase in depreciation more than offsets the drawback of a slower increase in gross production.

This effect can be quite considerable. In the case of $\alpha = 0.20$ and $r = 0.15$, an increase of T from $T_0 = 10$ to $T_1 = 20$ will cause ρ to increase from $\rho_0 = 3.6$ to $\rho_R = 5.4$ per cent, and an increase of T from $T_0 = 20$ to $T_1 = 30$ will cause ρ to increase from $\rho_0 = 3.8$ to $\rho_R = 4.5$ per cent. In the case of $\alpha = 0.20$ and $r = 0.20$, the same changes in T will cause ρ to increase from $\rho_0 = 4.9$ to 7.0 per cent and from 5.2 to 5.9 per cent, respectively.

4. CONCLUDING REMARKS

In Sections 2 and 3 we have obtained conclusions which may be summarized as follows: Consider first the case of a fixed gross investment quota. Comparing alternatives with the same internal rate of return to capital, we find that the durability of capital influences the asymptotic growth rate of production very considerably, with long durabilities corresponding to high growth rates. However, a switch-over from investment with a certain durability to investment with higher (or lower) durability does not immediately affect the growth rate of gross production very much.

In the case of the net model the conclusions are in a sense reversed. The asymptotic growth rate is then not much affected by the durability of capital when the internal rate of return is the same; whereas the immediate rate of growth of *net* national product will be affected strongly by a change in the durability of new investment under a constant internal rate of return. However, the immediate rate of growth of *gross* national product would not be affected more in this case than in the case of the gross model.

We might have worked out the solutions more completely, tracing out the time paths from a change in the durability until the asymptotic development corresponding to the new durability is approached. Such a complete solution is rather cumbersome since it would show points of discontinuities for the growth rates, and the solutions must be worked out separately for every interval between such points. Also, it might not be too interesting, since in real life all investments carried out in a certain period do not have the same durability, and this would tend to even out the discontinuities and differences between the intervals mentioned above.

The models employed in this paper are seemingly unrealistic since they include only capital as a factor of production. However, labor input could easily be taken into account in special cases.

Consider first the case of strict complementarity between labor and capital. If we assume real wages to be constant, we could introduce a coefficient expressing the labor cost per unit of product. We could then reduce ξ by this coefficient in defining the internal rate of return to capital. We would, however, obtain solutions which are formally similar to those above.

Another extreme assumption would be to adopt a production function which is linear in capital and labor input. If we assume labor input to be determined by exogenous factors, the solutions we have obtained above could be considered as components of the complete solution and would therefore still deserve interest.

The third growth factor, viz., technological progress, is more problematic. If existing capital equipment embodies the technological knowledge reached at the time of its production, the durability of capital will affect the speed with which new technological knowledge can be exploited. Our problem would then clearly be more complicated. Some theoretical suggestions concerning this aspect are given in my paper [4]. The relationship between the durability of capital and the exploitation of new techniques is analysed both theoretically and empirically by W.E.G. Salter [5].

University of Oslo, Norway.

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THE ELASTICITY OF DEMAND FOR IMPORTS WITH CLOSE, BUT IMPERFECT SUBSTITUTES*

BY W. M. GORMAN[1]

1. INTRODUCTION AND SUMMARY

A COUNTRY PRODUCES a substitute for the good it imports. Though the substitute is not a perfect one, the two goods are considerably more closely related to each other on the demand side than either is to any third good. The elasticity of substitution between them is θ and $100k$ per cent of their joint market is filled by the substitute, which is in fairly elastic supply. It is shown that the compensated elasticity of demand for imports is given by $\delta \doteq k\theta$, and that, if anything, this is an overestimate[2]. The argument in [1] that δ would be high and largely determined by the elasticity of supply of the import substitute is thus seen to break down badly when the substitute is an imperfect one. The danger that such apparently small specification errors may have large effects bedevils all of econometrics, and, indeed, of economic theory.

2. DERIVATION OF APPROXIMATION FORMULAS

A certain country exports e units of one good (E) in return for m of another (M). The latter is not produced locally, but s units of quite a close substitute (S) are. The domestic prices of M and S in terms of E are p and q, respectively, and the domestic demand for them is given by

$$(1) \qquad m = m(p, q, \cdots; u), \quad s = s(p, q, \cdots; u),$$

where u is a utility index[3]. The supply of the import substitute is

$$(2) \qquad s = f(q).$$

After compensated price changes,

$$(3) \qquad dm = (\partial m/\partial p)dp + (\partial m/\partial q)dq,$$

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[1] I am grateful to Professor L. R. Klein for pointing out the importance of this problem to me in a comment on [1], to Mr. E. Bennathan, Dr. F. H. Hahn and Professor D. C. Rowan for their comments on an earlier draft, and especially to the referee who pointed out several slips.

[2] δ approaches $k\theta$ asymptotically from below as the elasticity of supply of the domestic substitute tends to infinity.

[3] We assume the existence of a social indifference map.

$$ds = (\partial s/\partial p)dp + (\partial s/\partial q)dq = f'dq \,. \tag{4}$$

Moreover, the Slutsky equation implies that

$$\partial m/\partial q = \partial s/\partial p \,. \tag{5}$$

Equations (3), (4) and (5) are easily seen to imply that compensated elasticity of demand for imports, $\delta = -pdm/mdp$, is given by

$$\delta = \lambda - n\rho^2/(\sigma + \varepsilon) \,, \tag{6}$$

where

$$\lambda = -p\partial m/m\partial p, \quad \rho = q\partial m/m\partial q \tag{7}$$

are the elasticities of demand for the import good with respect to its own and its substitute's prices,

$$\varepsilon = -q\partial s/s\partial q, \quad \sigma = qf'/s \tag{8}$$

are the elasticities of demand and supply of the import substitute, and

$$n = pm/qs \tag{9}$$

measures the importance of M compared with S.[4]

Suppose next that M and S are considerably more closely related to each other than either is to any third good. Then the budget constraint may be written

$$pm + qs + \Sigma r_i x_i = y \,,$$

where the x's are the quantities of the other goods consumed at home; the r's are their prices; and y is income. If p and q change by dp and dq the compensating income change is

$$dy = mdp + sdq \,, \tag{10}$$

neglecting induced changes in the other prices[5]; therefore

$$(p\partial m/\partial p + q\partial s/\partial p)dp + (p\partial m/\partial q + q\partial s/\partial q)dq \doteq 0 \,. \tag{11}$$

Since dp and dq are arbitrary, (3), (4), (5) and (11) imply that

$$m(\lambda - \rho) \doteq s(n\rho - s) \doteq 0 \,; \tag{12}$$

therefore

[4] All demand elasticities are computed with u constant. That is, they are compensated elasticities.

[5] This seems reasonable because of our assumption that M and S are considerably more closely related to each other than either is to any third good. Actually we could have taken account of the interactions at this stage without affecting the results. The referee has pointed out to me that Hicks [2 (315–17)] and Mosak [4 (105–9)] have discussed related problems.

(13) $$\lambda \doteq \rho \doteq \varepsilon/n .$$

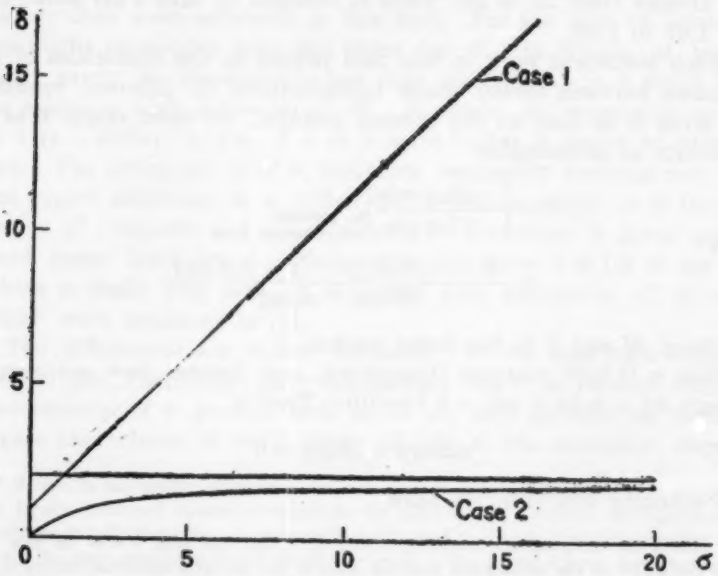
Substituting into (6), we find that

(14) $$\delta \doteq \lambda(1 + n\lambda/\sigma)^{-1} .$$

Hence $\delta \leqq \lambda$ and approaches it asymptotically as $\sigma \to \infty$. There is a good deal of evidence that demand elasticities tend to be quite low even in cases where we would say that close substitutes exist. Some cases will be noted below. Thus δ seems likely to be low. On the other hand, I argued in [1] that supply tends to be highly elastic in the long run. If this is so in the case of the country which we have been discussing, and if the locally produced substitute is an important one, so that n is not large, $n\lambda/\sigma$ will be quite small, and $\delta \doteq \lambda$. The elasticity of supply of the locally produced substitute therefore plays quite a minor part in determining the elasticity of demand for imports.

In [1] it was assumed that the substitute was a perfect one, so that $\lambda = \infty$. In that case it was found that

(15) $$n\delta = (n + 1)\mu + \sigma ,$$



Case 1: $\lambda = \infty$, $\mu = 0.5$, $\delta_1 = \sigma + 1$, $n = 1$
Case 2: $\lambda = 2$, $\delta_2 = 2\sigma/(\sigma+2)$, $n = 1$

FIGURE 1: RELATIONSHIP BETWEEN δ AND σ

where μ was the elasticity of demand for the import good[6]. Given that σ was likely to be large, it was argued that δ would be large too, and would be mainly determined by σ. All this is now seen to have depended critically on the assumption that the substitute was a perfect one.

The relationship between these two results can most easily be seen with the aid of Figure 1, which compares two cases. In one, M and S are prefect substitutes, so that $\lambda = \infty$ (and $\mu = 0.5$, which is quite reasonable), while, in the other, they are imperfect substitutes, and $\lambda = 2$. In each case, $n = 1$ so that the domestic expenditure on the import good is the same as that on its substitute. In the second case δ approaches $\lambda = 2$ asymptotically as $\sigma \to \infty$, with the result that the curve flattens out quite soon. In the first case it approaches $\lambda = \infty$ asymptotically in a sense, but never has to flatten.

We notice the following points:

1. In the first case δ is much greater. For instance, $\delta_1 = 11$ and $\delta_2 = 1.67$ when $\sigma = 10$.
2. In the second case changes in σ are much less important. For instance, δ_1 increases by 90 per cent from 11 to 21 when σ increases from 10 to 20, while δ_2 changes by only 9 per cent—from 1.67 to 1.82.

Much statistical work in this field relates to the elasticities of substitution between similar goods manufactured in different countries. To bring it to bear on the present problem, we must relate λ to the elasticity of substitution

$$\theta = \left[\frac{-\delta \log m/s}{\delta \log p/q}\right]_{\substack{u,\ \text{income,} \\ \text{other prices fixed}}} = \frac{(\lambda + n\rho)d\log p - (\varepsilon + \rho)d\log q}{d\log p - d\log q} \tag{16}$$

between M and S in the home market.

Now u is held constant throughout, and income does not change[7]. Hence $dy = mdp + sdq = 0$ by (10). That is

$$n d\log p + d\log q = 0 . \tag{17}$$

Substituting into (16), we have

[6] This is the compensated elasticity, of course. Note, too, that $\lambda = \infty$ in this case. μ is computed on the assumption that the price of the (perfect) substitute moves in step with that of the imports themselves.

[7] Note that the other prices, the r's, are treated as constant because the demand for the corresponding goods is assumed to be little affected.

(18) $$\theta = (\lambda + n\varepsilon + 2n\rho)/(n + 1)\,.$$

Reverting to the assumption that M and S are notably better substitutes for each other than either is for any other good, we can substitute from (12) and (13) into (18) to find

(19) $$\lambda \doteq k\theta$$

and

(20) $$\delta \doteq k\theta[1 + (1 - k)\theta/\sigma]^{-1}$$

by (14), where

(21) $$k = 1/(n + 1) = qs/(pm + qs)$$

measures the importance of the substitute in the total market for it and the import good.

3. SOME EMPIRICAL IMPLICATIONS

The first thing to notice about these results is that $\delta \leqq k\theta$, which it approaches asymptotically as $\sigma \to \infty$. Sir Donald MacDougall [3] has estimated that the elasticity of substitution between British and American exports in third markets is about 3, which is a good deal higher than most estimates in this field. For the sake of illustration we might reasonably take this value for θ[8]. In Britain at least, k must surely be considerably less than one-half, while 6 would be by no means a high estimate of σ[9]. It might be reasonable to take $[1 + (1 - k)\theta/\sigma]^{-1} = 3/4$; if σ or k were higher it would be closer to one. The estimate $\delta \doteq k\theta$ is, therefore, reasonably accurate only when the import substitute is in rather highly elastic supply or is the main source of "imports and quasi-imports." However, it gives quite a good upper limit for δ. Notice that this gives $\delta \leqq 1.5$ in our case, which is really very low. It compares with elasticities of 10 or 20 which were predicted in [1].

The differences are rather dramatic. If we took this extremely limited model seriously, their implications would be equally startling. The removal of a general tariff at 20 per cent *ad valorem* would increase the volume of world trade 13-fold if the elasticity were 15,

[8] There is no good reason why the elasticity of substitution between an import and its locally produced substitute should be the same as that between the exports of two countries in a third market. I personally would tend to expect it to be somewhat lower. If so, the main argument would be strengthened. The actual figures are intended only for illustration.

[9] Unless one were to include distant substitutes in S, which would drive θ down and also render the underlying assumptions invalid.

by only 30 per cent if it were 1.5. Relatively small price changes would right exchange imbalances, at least in the long run, were the elasticity high. Unemployment and trade restrictions would be likely if it were low. Needless to say, the model is altogether too unrealistic for us to draw such conclusions; nevertheless the effects of dropping the "perfect substitute" assumption are remarkable, and suggest that we should look very closely at such assumptions elsewhere. What, in particular, happens to the theory of factor price equalization if we drop that assumption?

University of Birmingham, U.K.

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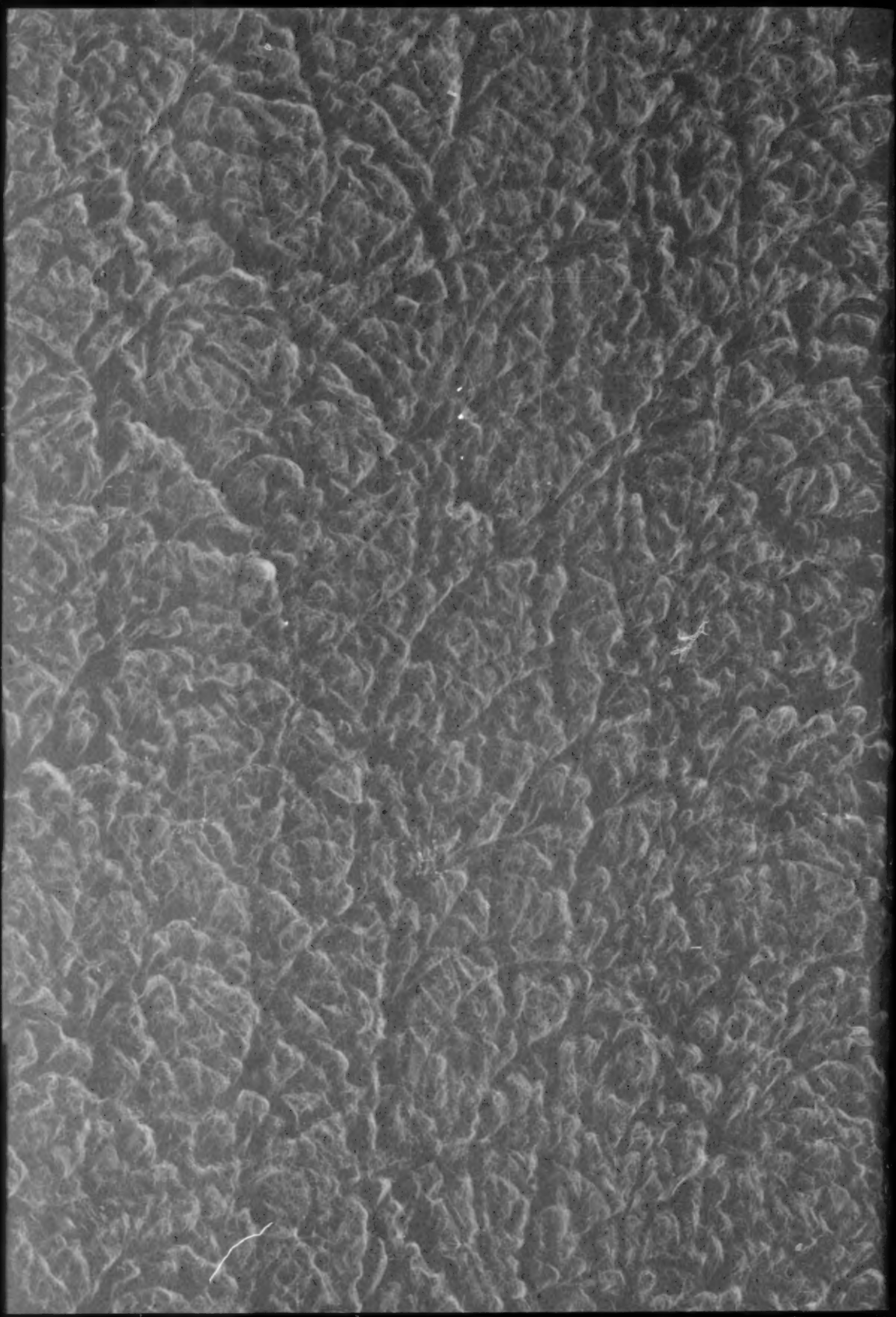
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